# Conjugated heat transfer in a package of fins

A. Lehtinen & R. Karvinen
*Institute of Energy and Process Engineering,*
*Tampere University of Technology, Finland*

## Abstract

The conjugated convection-conduction heat transfer problem has been solved for a package of fins used in the cooling of electronics. Both laminar and turbulent forced convection flows have been considered. Most of the results published previously have neglected conduction resistance. The solution procedure used in this paper is semi-analytic. The conduction in the fins is assumed to be only in the direction normal to the fin base. The convective heat transfer is modelled assuming fully developed velocity profile in the laminar case and constant heat transfer coefficient in the turbulent case. The conduction in the fluid is neglected in directions parallel to fins. With the above-mentioned simplifications, partial differential equations can be Laplace transformed to obtain an ordinary differential equation. Finally, total heat flux can be achieved by inverse Laplace transforming a resulting series expansion. The results obtained in this paper can be used to obtain optimum plate spacing and corresponding heat flux for a given mass of fin material, pumping power and Prandtl number. The results are compared to existing results for isothermal fins.
*Keywords: conjugated heat transfer, electronics cooling, forced convection in channels, fin package, plate spacing, optimisation, fixed pumping power.*

## 1 Introduction

The basic element in many electronics cooling applications is a rectangular fin package cooled by forced convection, shown in Figure 1. The most important issues for the designer are the optimal fin spacing and the corresponding obtained heat flux. These issues have been explored to some extent by many authors for isothermal or uniform heat flux plates, for example by Bejan and Sciubba [1], Mereu *et al.* [2] and Campo and Li [3].

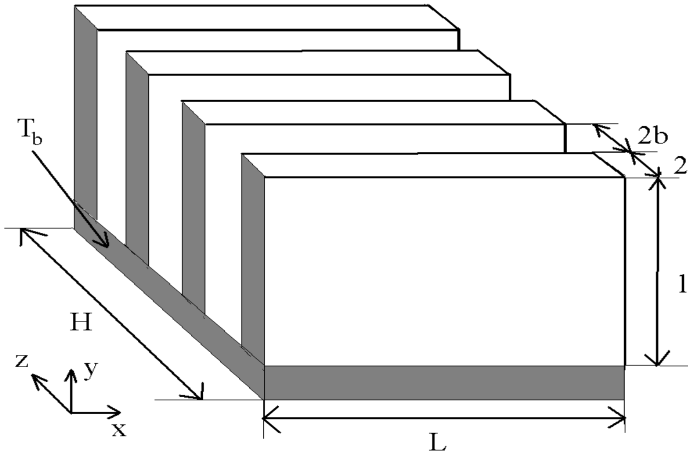


Figure 1: Schematic picture of the rectangular fin package.

However, if the amount of fin material is to be optimised, the fins can rarely be considered isothermal. Karvinen [4] has analysed conjugated convection-conduction heat transfer in a single fin in free stream. The purpose of this paper is to provide the designer with fin package performance results when the fin efficiency is below 1.

The optimal fin spacing and especially the corresponding heat flux depend on the flow conditions. As stated by Mereu *et al.* [2], there are three typical flow conditions: fixed mass flow rate, fixed pressure drop and fixed pumping power. The emphasis in this paper is in the most realistic and the most difficult case of fixed pumping power. However, the methods presented in this paper can also be used to obtain results for the other flow conditions.

## 2 Governing equations

The fin half-thickness $t$ is usually very small compared to the other fin dimensions $L$ and $l$. Thus, the fin temperature can be assumed to be uniform in the $z$ direction. Furthermore, the temperature gradient is assumed to possess a much larger component in y direction (normal to the fin base) than in $x$ direction (parallel to flow), except near the leading edge [5]. For simplicity of analysis, only conduction in $y$ direction is taken into account in the energy equation to yield

$$k_s t \frac{\partial^2 T(x,y)}{\partial y^2} = q(x,y), \tag{1}$$

where $q(x,y)$ is the heat flux from the fin to the fluid. In the turbulent case, the convective heat transfer coefficient may be assumed to be constant with only a slight error. The following simple correlation that is applicable for gases is used [6].

$$q(x,y) = h(T - T_m) = \frac{0.021 k_f}{D_h} \Pr^{0.5} \mathrm{Re}_{D_h}^{0.8} (T - T_m). \tag{2}$$

The result of eqn (2) is obtained assuming constant surface temperature, but may be used as an approximation for axially varying surface temperature. The laminar case is much more difficult because the heat transfer coefficient diminishes rapidly in streamwise direction and is dependent on the whole upstream fin temperature distribution. In this paper, analysis is simplified by assuming fully developed velocity distribution (large Prandtl number). Furthermore, conduction in the fluid is neglected in directions parallel to the fin. In other words, the solution to the Graetz problem with arbitrarily varying wall temperature is used to model *q(x,y)* [6]. Since $b << l$, the results for parallel plates can be used:

$$q(x,y) = \frac{k_f}{b} \sum_{n=0}^{\infty} G_n \exp(-\lambda_n^2 x^+) \left(T(0,y) - T_0\right) + \frac{k_f}{b} \int_{\xi=0}^{x^+} \sum_{n=0}^{\infty} G_n \exp\left(-\lambda_n^2 (x^+ - \xi)\right) \frac{dT(\xi,y)}{d\xi} d\xi, \tag{3}$$

where $x^+ = x/(2b\mathrm{Re}_{Dh}\mathrm{Pr})$. The eigenvalues $\lambda_n$ and the corresponding eigenfunctions $G_n$ are given in Table (1).

Table 1: Graetz function eigenvalues and eigenfunctions.

| n | $\lambda_n$ | $G_n$ |
|---|---|---|
| 0 | 3.884 | 1.717 |
| 1 | 13.09 | 1.139 |
| 2 | 22.32 | 0.952 |
| >2 | 9.237n+3.849 | $2.68\lambda_n^{-1/3}$ |

The assumption of fully developed velocity profile somewhat underestimates heat transfer, but the effect is relatively small for $b << L$ even in the Prandtl number range of gases.

As we are mostly dealing with the case of fixed pumping power, a model for the pressure drop across the fin package is needed. In this paper, entrance and exit losses are neglected and the friction losses are calculated assuming fully developed flow [6]. The pressure drop is given by

$$\Delta p = \frac{c_f}{2} \rho V^2 \frac{L}{b}, \tag{4}$$

where

$$c_f = \frac{24}{\mathrm{Re}_{D_h}}, \qquad \text{laminar flow,} \tag{5}$$

$$c_f / 2 = 0.023 \mathrm{Re}_{Dh}^{-0.2}, \text{ turbulent flow, } 3*10^4 < \mathrm{Re} < 10^6. \tag{6}$$

The mass flow rate through the whole fin package is, for $H >> (b+t)$,

$$\dot{m} = \frac{\rho V l H b}{b+t}. \tag{7}$$

An important parameter in the field of forced convection fin cooling is the non-dimensional pumping power $\Phi$, as defined by Mereu *et al.* [2]:

$$\Phi = \frac{\dot{m}\Delta p L^3}{H l \mu^2 \nu}. \tag{8}$$

The definition of $\Phi$ has the advantage that it is relatively constant for a given fan (or pump) and fin package outer dimensions, varying only due to the fan efficiency dependence on the operating point.

## 3 Solution for turbulent flow

Using eqns (1) and (2), the governing differential equation and the boundary conditions for the fin temperature is

$$k_s t \frac{\partial^2 T(x,y)}{\partial y^2} = h\big(T(x,y) - T_m(x,y)\big),$$
$$T(x,0) = T_b, \tag{9}$$
$$\frac{\partial T(x,l)}{\partial y} = 0,$$

and for the fluid temperature

$$\rho c_p b V \frac{\partial T_m(x,y)}{\partial x} = h\big(T(x,y) - T_m(x,y)\big),$$
$$T_m(0,y) = T_0. \tag{10}$$

Using the non-dimensional variables $X=x/L$, $Y=y/l$, $\theta(x,y)=(T(x,y)-T_0)/(T_b-T_0)$, $A=k_s tL/(\rho c_p bVl^2)$, $B=hL/(\rho c_p bV)$ and Laplace transforming eqns (9) and (10) with respect to $X$ yields

$$\frac{d^2\Theta(s,Y)}{dY^2} = \frac{B}{A}\frac{s\Theta(s,Y)}{s+B},$$
$$\Theta(s,0) = \frac{1}{s}, \tag{11}$$
$$\frac{\partial\Theta(s,1)}{\partial Y} = 0,$$

where $\Theta(s,Y)=L_X\{\theta\}$ is the Laplace transformed non-dimensional fin temperature. Solving eqn (11) and using the boundary conditions results in

$$s\Theta(s,Y) = \cosh\left(\sqrt{\frac{Bs}{As+AB}}Y\right) - \tanh\left(\sqrt{\frac{Bs}{As+AB}}\right)\sinh\left(\sqrt{\frac{Bs}{As+AB}}Y\right). \tag{12}$$

The fin temperature distribution *T(x,y)* could be found by inverse Laplace transforming *Θ(s,Y)* in eqn (12). However, the inversion can hardly be performed analytically. Luckily, in our application the exact temperature distribution is not desired, but rather the total heat flux transferred by the fin package. Since the total heat flux may be written as

$$Q = \frac{H}{b+t}\int_0^L -k_s t \frac{\partial T(x,0)}{\partial y} dx, \tag{13}$$

and the partial fraction expansion for hyperbolic tangent [7] as

$$\tanh(x) = \sum_{j=0}^{\infty} \frac{8x}{(2j+1)^2 \pi^2 + 4x^2}, \tag{14}$$

an analytical inverse Laplace transform may be found to yield

$$\frac{Q}{\dot{m}c_p(T_b - T_0)} = 1 - 8\sum_{j=0}^{\infty} \frac{\exp\left(-\frac{(2j+1)^2\pi^2 AB}{(2j+1)^2\pi^2 A + 4B}\right)}{(2j+1)^2\pi^2}. \tag{15}$$

For isothermal fins ($A=\infty$, $B=NTU$) eqn (13) reduces to the familiar form

$$\frac{Q}{\dot{m}c_p(T_b - T_0)} = 1 - \exp(-NTU). \tag{16}$$

## 4 Solution for laminar flow

Using eqns (1) and (3), Laplace transforming with respect to $x^+$ and using the same boundary conditions as in eqn (9) results in

$$s\Theta(s,Y) = \cosh\left(\sqrt{F}Y\right) - \tanh\left(\sqrt{F}\right)\sinh\left(\sqrt{F}Y\right), \tag{17}$$

where

$$F = \sum_{n=0}^{\infty} \frac{G_n s}{\kappa(s + \lambda_n^2)} \tag{18}$$

and $\kappa = k_s tb/(k_f l^2)$ is the non-dimensional fin conductivity.

As with turbulent flow, analytical inverse Laplace transforming *Θ(s,Y)* is hardly possible and actually not necessary. Using eqns (13) and (14), the total heat flux transferred by the fin package may be written as

$$\frac{Q}{\dot{m}c_p(T_b - T_0)} = 64 L_{x^+}^{-1}\left\{\frac{1}{s^2}\sum_{j=0}^{\infty}\left(\frac{4}{\kappa} + \frac{(2j+1)^2\pi^2}{\sum_{n=0}^{\infty}\frac{G_n s}{s+\lambda_n^2}}\right)^{-1}\right\}. \tag{19}$$

No closed-form inverse Laplace transform for the expression in eqn (19) is known to the authors. However, the solution is possible if both of the infinite summations are truncated and approximated by a finite number of terms.

 
www.witpress.com, ISSN 1743-355X (on-line)

$$\frac{Q}{\dot{m}c_p(T_b - T_0)} \approx 64L_{x^+}^{-1}\left\{\frac{1}{s^2}\sum_{j=0}^{J}\left(\frac{4}{\kappa}+\frac{(2j+1)^2\pi^2}{\sum_{n=0}^{N}\frac{G_n s}{s+\lambda_n^2}}\right)^{-1}\right\}. \tag{20}$$

For a large number of terms, the inverse Laplace transform in eqn (20) is very complicated but may be found with any standard symbolic mathematics software for a given non-dimensional fin conductivity $\kappa$, the number of Graetz function eigenvalues $N$ and the number of hyperbolic tangent partial fraction expansion terms $J$.

## 5 Optimal plate spacing and corresponding heat flux

The results obtained for total heat fluxes for turbulent flow (as a function of $A$ and $B$) and laminar flow (as a function of $x^+$ and $\kappa$) have a certain value of their own. In practice, however, one is often interested in the optimal plate spacing and the corresponding heat flux as functions of non-dimensional pumping power $\Phi$ and non-dimensional fin mass $K=k_s tL^2/(k_f bl^2)$. Approximate results for isothermal fins ($K=\infty$) have been previously obtained with the method of intersecting asymptotes [2]. The results were developed for laminar flow, but they can be generalised to be applicable for both laminar and turbulent flows using different constants for different types of flows. The results for the method of intersecting asymptotes are

$$\left(\frac{b}{L}\right)_{opt}\frac{\Pr^{\alpha}\left[\Phi\left(1+\frac{t}{b}\right)\right]^{\beta}}{C_1}\approx 1, \tag{21}$$

$$\frac{Q_{opt}L}{C_2 H k_f \Delta T l}\Pr^{-\chi}\Phi^{-\lambda}\left(1+\frac{t}{b}\right)^{\gamma}\leq 1, \tag{22}$$

where the constants $C_1$, $C_2$, $\alpha$, $\beta$, $\chi$, $\lambda$ and $\gamma$ are different for laminar and turbulent flows and are tabulated in Table (2).

Table 2: Constants in eqns (21) and (22).

| | $C_1$ | $C_2$ | $\alpha$ | $\beta$ | $\chi$ | $\lambda$ | $\gamma$ |
|---|---|---|---|---|---|---|---|
| Turbulent | 0.0352 | 1.0126 | 7/20 | 1/16 | 17/20 | 37/112 | 75/112 |
| Laminar | 1.1311 | 0.6530 | 10/27 | 1/6 | 17/27 | 1/3 | 2/3 |

In this paper, we are interested in the non-isothermal case. Using the friction loss model of eqns. (4)-(6), the mass flow equation (7) and the definition of $\Phi$ in eqn (8), the total heat flux results of eqns (15) and (20) can easily be written as functions of $\Phi$, $K$, $Pr$, $t/b$ and $b/L$:

$$\frac{QL\left(1+\frac{t}{b}\right)^{\frac{2}{3}}}{Hk_f \Delta Tl \, \mathrm{Pr}^a \, \Phi^{\frac{1}{3}}} = \varepsilon\left(\frac{b}{L}\mathrm{Pr}^b\left[\left(1+\frac{t}{b}\right)\Phi\right]^c, \mathrm{Pr}^{-a}\left[\left(1+\frac{t}{b}\right)\Phi\right]^{-\frac{1}{3}}K\right), \quad (23)$$

where the constants $a$, $b$ and $c$ are different for laminar and turbulent flow and are tabulated in Table (3). The non-dimensional total heat flux function $\varepsilon$ has a unique maximum with respect to *b/L*, although its analytical determination is not possible. However, it is rather straightforward to determine the maximum and the corresponding total heat fluxes numerically for any value of the latter argument of the function $\varepsilon$ in eqn (23). In other words, the optimal b/L may be found numerically for any set of $\Phi$, *K, Pr* and *t/b*, all of which are assumed to be known before the optimisation. Any standard numerical maximisation method, such as the Gauss-Newton algorithm, can be used since the function $\varepsilon$ has only a single extremum.

After neglecting streamwise conduction in the fin and normal-to-plate velocity in the fluid, it cannot be claimed that the method described above would give more accurate results than eqn (21), or would at least be worth the expense. However, in the fixed pumping power case, which we are mainly dealing with, the total heat flux is relatively constant in the vicinity of the maximum. Thus, eqn (21) is quite satisfactory.

While the dependence of the total heat flux on *b/L* is relatively small, the contrary is true of its dependence on $\Phi$ or *K*. As noted previously, the assumptions of fully developed velocity profile and negligible conduction in streamwise direction somewhat underestimate the total heat flux. Thus, eqn (23) gives the lower limit of the heat flux as eqn (22) gives the upper limit. The real heat flux is expected to be much closer to the lower limit, especially for small *K*.

Table 3: Constants in eqns (23), (24) and (25).

| | a | b | c | $C_3$ | $C_4$ |
|---|---|---|---|---|---|
| Turbulent | 5/6 | 7/18 | 1/18 | 0.0200 | 0.6765 |
| Laminar | 2/3 | 1/3 | 1/6 | 1.2319 | 0.5941 |

## 6 Results

The optimal plate spacing *b/L* was found numerically for several combinations of $\Phi$, *K, Pr* and *t/b*. The optimisation was performed using the Gauss-Newton algorithm. In the turbulent case, the infinite summation in eqn (15) was approximated by 2000 terms. The computational effort increased linearly with respect to the number of terms used.

In the laminar case, the numbers of summation terms in eqn (20) were *J=200* and *N=8*. The computational effort increased linearly with *J* but approximately as *exp(0.6*N)* with *N*. Increasing the number of terms from these values was shown to have negligible effect on the results.


www.witpress.com, ISSN 1743-355X (on-line)

As suggested by eqn (23), the optimal plate spacing can be presented by a single curve when suitable non-dimensional parameters are used. The results are shown in Figure 2 for both turbulent and laminar flow. It is seen that in both turbulent and laminar cases the optimal plate spacing behaves irregularly as a function of *K*. However, the optimal plate spacing is nearly constant, being only slightly below the isothermal value.

Taking into account the approximations introduced in the current analysis, it does not seem reasonable for the designer to use the results in Figure (2). Instead, the easily usable isothermal limit of the current results is recommended:

$$\left(\frac{b}{L}\right)_{opt} \mathrm{Pr}^{b}\left[\left(1+\frac{t}{b}\right)\Phi\right]^{c} = C_3, \tag{24}$$

where the constants *b*, *c* and $C_3$ are given in Table (3). In the majority of cases eqn (24) gives results of the same order of magnitude as eqn (21), which was obtained with the method of intersecting asymptotes.

More interesting are the maximum total heat flux results that were obtained with the numerical optimisation procedure. In the isothermal case ($K=\infty$), the maximum heat flux was found to be given by the equation

$$\frac{Q_{\max,isot} L\left(1+\frac{t}{b}\right)^{\frac{2}{3}}}{H k_f \Delta T l \,\mathrm{Pr}^{a}\, \Phi^{\frac{1}{3}}} = C_4, \tag{25}$$

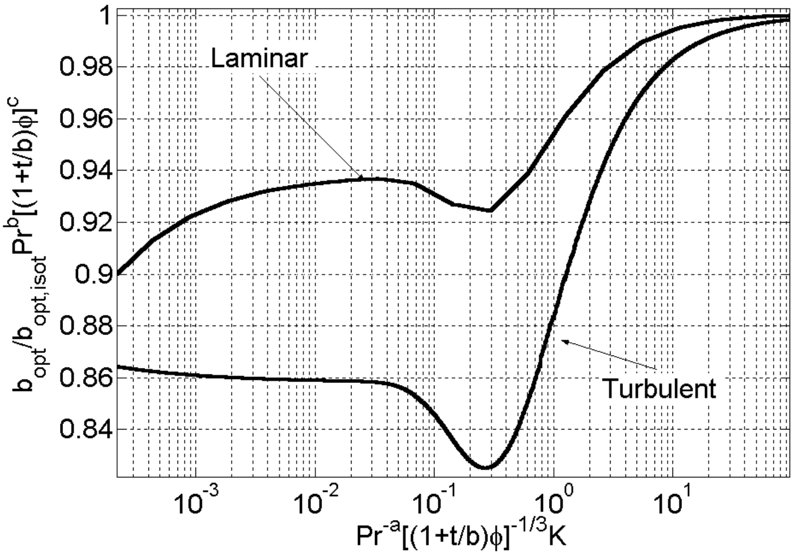


Figure 2: Optimal plate spacing ratio to isothermal optimum as a function of non-dimensional fin mass.

where the constants $a$ and $C_4$ are given in Table (3). Due to its asymptotic nature, eqn (22) gives considerably larger heat fluxes than eqn (25). The numerically computed fin efficiencies in the optimal conditions are shown in Figure 3 as functions of the non-dimensional fin mass. Another way to achieve the approximate fin efficiencies is to assume a constant heat transfer coefficient with respect to the ambient temperature. The mean heat transfer coefficient is obtained from eqn (25):

$$\bar{h} = \frac{Q_{\max,isot}(t+b)}{\Delta TLlH} = \frac{C_4(t+b)\,\mathrm{Pr}^a\,\Phi^{\frac{1}{3}}k_f}{L^2\left(1+\frac{t}{b}\right)^{\frac{2}{3}}}. \tag{26}$$

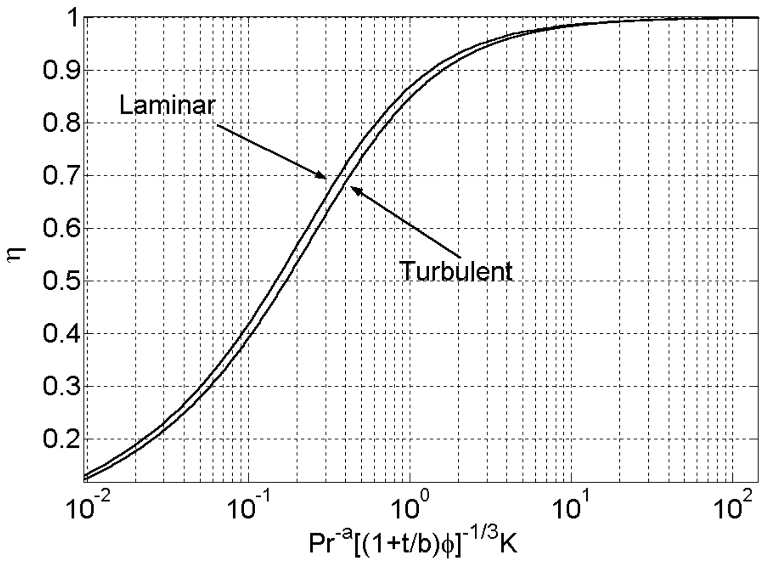


Figure 3: Fin efficiency as a function of non-dimensional fin mass.

Following with the conventional fin theory, the fin efficiency can be written as

$$\eta = \frac{\tanh\left(\sqrt{C_4}\,\mathrm{Pr}^{\frac{a}{2}}\left[\Phi\left(1+\frac{t}{b}\right)\right]^{\frac{1}{6}}K^{-\frac{1}{2}}\right)}{\sqrt{C_4}\,\mathrm{Pr}^{\frac{a}{2}}\left[\Phi\left(1+\frac{t}{b}\right)\right]^{\frac{1}{6}}K^{-\frac{1}{2}}}. \tag{27}$$

Eqn (27) gives fin efficiencies somewhat below those shown in Figure 3, but the difference is not greater than 5 %.

## 7 Conclusions

Analytical solutions for both turbulent and laminar conjugated heat transfer in a package of fins were obtained. As an application, the results were used to optimise the plate spacing in the realistic case of fixed pumping power. The results obtained for the maximum total heat transfer rates can be very useful in the design process of a rectangular fin heat exchanger. Also analytical expressions for the optimal plate spacing and the corresponding heat flux were obtained. It was shown that the analytical expressions provide results very close to those obtained with numerical optimisation procedure.

## References


[1] Bejan, A. & Sciubba, A., The optimal spacing of parallel plates cooled by forced convection. *Int. J. Heat Mass Transfer*, **35(12)**, pp. 3259-3264, 1992.

[2] Mereu, S., Sciubba, E. & Bejan, A., The optimal cooling of a stack of heat generating boards with fixed pressure drop, flowrate or pumping power. *Int. J. Heat Mass Transfer,* **36(15)**, pp. 3677-3686, 1993.

[3] Campo. A & Li, G., Optimum separation of asymmetrically heated sub-channels forming a bundle: influence of simultaneous flow and temperature, *Heat and Mass Transfer,* **32**, pp. 127-132, 1996.

[4] Karvinen, R., Efficiency of straight fins cooled by natural or forced convection. *Int. J. Heat Mass Transfer,* **26(4)**, pp. 635-638, 1983.

[5] Karvinen, R., Natural and forced convection heat transfer from a plate fin, *Int. J. Heat Mass Transfer,* **24**, pp. 881-885, 1981.

[6] Kays, W.M. & Crawford, M.E., *Convective Heat and Mass Transfer,* McGraw-Hill, Inc: New York, 1993.

[7] Spanier, J. & Oldham, K.B., *An Atlas of Functions*, Hemisphere Publishing Corporation, 1987.