Dynamics of prestressed beams coupled with a string

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Abstract

A new coupled system beam and string, connected together with an elastic layer of Winkler type, is suggested in this paper. The double system presents the diminishing of the response under a moving force in some cases. The theoretical model looks like a Bernoulli-Euler beam with an axial force and a string with a tensile force. The investigated case represents a simple theoretical model of a prestressed bridge.

Keywords: complex system beam and string, moving loads, prestressed bridges.

1 Motivation

The engineers have tried to diminish the dynamic response of structures for a long time. The elastic bearings or elastic supports provide one of the possibilities which can damp the induced vibrations in some cases. They are applied in seismic regions, e.g. in East Asia, [1]. Recently, the triangular falsework has been investigated, [2], for damping and controlling the vibration – a type which was used in the past time.

The prestressed beams have often been applied to prestressed bridges where the reinforced concrete beams are stressed by the pretensile strings. The behaviour of such a system do not affect too much their dynamics excepting that the natural frequencies are a little reduced, [3].

On the other hand, several authors have investigated the double system in recent time : two beams, [4] to [7, 11, 12], or two strings, [8] to [10], connected together with an elastic layer of Winkler type. The double system diminishes the amplitudes of vibration in some cases and, thus, brings some advantages.
However, the application of double system mentioned above can hardly be found in practice of bridge engineering. The reasons are: the high price in the first case (double beam) or their application to long spans only in the second case (double string). The second system is too soft and, thus, not applicable to short and medium spans which may be frequently found in both the railway and highway lines.

Consequently, a new idea arose – to join a beam and a tensile string with an elastic layer and to form another double system. The tensile force in the string is, of course, applied to the beam as a stress force.

The main aim of the paper is to discover such a combination of input parameters for which the response of the beam may be lower than its static one. It may happen at the expense of the string, of course. Nevertheless, the complex system could contribute to the diminishing of dynamic effects and, thus, to their safety and economy.

2 Simple theoretical model

The simplest theoretical model in bridge engineering is represented by a Bernoulli-Euler beam of span \( l \) subjected to a force \( F \) moving at speed \( c \), see Fig. 1, [13]. The string of the same span \( l \) is connected to the beam with an elastic layer of Winkler type with the characteristics \( k \). The beam is generally tensed by the force \( N_1 \) while the string by the force \( N_2 \). The Fig. 1 shows the positive (tensile) directions of both forces. In practice, of course, may be \( N_1 = -N_2 \).

The following set of the coupled partial differential equations describes the behaviour of both the beam and string

\[
EI \frac{\partial^4 v_1(x, t)}{\partial x^4} - N_1 \frac{\partial^2 v_1(x, t)}{\partial x^2} + \mu_1 \frac{\partial^2 v_1(x, t)}{\partial t^2} + k[v_1(x, t) - v_2(x, t)] = \varepsilon(t) \delta(x - ct) F, \tag{1}
\]
\[-N_2 \frac{\partial^2 v_2(x, t)}{\partial x^2} + \mu_2 \frac{\partial^2 v_2(x, t)}{\partial t^2} + k[v_2(x, t) - v_1(x, t)] = 0,\]  
(2)

where it is denoted:

- \(v_1(x, t)\) or \(v_2(x, t)\) – vertical deflection of the beam or string, respectively, at \(x\) and time \(t\),
- \(EI\) – constant bending stiffness of the beam,
- \(N_1\) or \(N_2\) – tensile forces in the beam or string, respectively,
- \(\mu_1\) or \(\mu_2\) – constant mass per unit length of the beam or string, respectively,
- \(k\) – spring characteristics of the Winkler foundation, [N/mm\(^2\)],

\[\varepsilon(t) = h(t) - h(t - l/c)\]  
(3)

– a function which expresses the presence or absence of the force on the beam,

\[h(t) = \begin{cases} 
0 & \text{for } t < 0 \\
1 & \text{for } t \geq 0 
\end{cases}\]  
(4)

– Heaviside unit function,

\(\delta(x)\) – Dirac delta function describing the single concentrated force.

The boundary and initial conditions of both the simply supported beam and string as well as the zero initial conditions are supposed. The methods of integral transformations were applied to the solution of equations (1) and (2), [14]. The deflection-time histories are derived and published in [15].

The following notations will be used: \(j = 1, 2, 3, \ldots\), and

\[
\omega_j^2 = \frac{j^4 \pi^4}{l^4} \frac{EI}{\mu_1}, \quad f_j = \frac{\omega_j}{2\pi}
\]  
(5)

for the natural frequency of vibration of a simple beam,

\[
\omega_{1j}^2 = \frac{j^4 \pi^4}{l^4} \frac{EI}{\mu_1} + \frac{j^2 \pi^2}{l^2} \frac{N_1}{\mu_1}
\]  
(6)

for the frequency of a beam with an axial tensile force \(N_1\),

\[
\omega_{2j}^2 = \frac{j^2 \pi^2}{l^2} \frac{N_2}{\mu_2}
\]  
(7)

for the frequency of a string tensed by the force \(N_2\),

\[
\omega_{1k}^2 = \frac{k}{\mu_1}, \quad \omega_{2k}^2 = \frac{k}{\mu_2}
\]  
(8), (9)

\[
\omega_{1jk}^2 = \omega_{1j}^2 + \omega_{1k}^2, \quad \omega_{2jk}^2 = \omega_{2j}^2 + \omega_{2k}^2
\]  
(10), (11)

\[
\omega = \frac{\pi c}{l}
\]  
(12)

for the exciting frequency implied by the moving force.
The following notations will be further used:

\[ v_0 = \frac{2F}{\mu_1 l \omega_1^2} \]  

(13)

\[ \alpha = \frac{c}{2f_1 l} \]  

(14)

– for the deflection of the beam center due to the force \( F \), applied at the same point,

\[ \Omega_{1,2} = \frac{\omega_{1j}^2 + \omega_{2j}^2}{\omega_1^2}, \quad A_1^2 = \frac{\Omega_1^2}{\omega_1^2}, \quad A_2^2 = \frac{\Omega_2^2}{\omega_1^2}, \]  

(15), (16)

\[ B_1^2 = \frac{\omega_{2j}^2}{\omega_1^2}, \quad B_2^2 = \frac{\omega_{2k}^2}{\omega_1^2}. \]  

(17), (18)

The natural frequencies of the coupled system beam + string are going after putting zero to the denominator of the solution, [15], [16]:

\[ \Omega_{1,2}^2 = \frac{1}{2}(\omega_{1j}^2 + \omega_{2j}^2) \pm \left[ \frac{1}{4}(\omega_{1j}^2 - \omega_{2j}^2)^2 + \omega_{1k}^2 \omega_{2k}^2 \right]^{1/2}, \quad \Omega_1 < \Omega_2. \]  

(19)

The special cases were also derived in [15]: statics (\( \alpha = 0 \)) and for critical speeds which appear at \( j\omega = \Omega_i, i = 1, 2 \). The numerical analysis was performed in the following range of dimensionless parameters: \( \alpha = 0 \) to 2, \( A_1^2 = 0.4 \) to 1, \( A_2^2 = 1 \) to 1.6, \( B_1^2 = 0.8 \) to 1.4, and \( B_2^2 = 0.2 \) to 0.8 with a step of 0.2. Altogether 2816 cases. The samples of the deflection-time histories for various speeds can be seen in [15].

3 Effect of some parameters

3.1 Effect of the speed

The movement of the force \( F \) along the beam is characterized by the dimensionless parameter \( \alpha \), Eq. (14). The maximum values \( \max v_1 \) of the beam as well as of the string \( \max v_2 \) roughly rise with increasing speed as can be seen in Fig. 2. The tendency reaches its maximum at about \( \alpha \leq 1 \), then it diminishes. The calculations get out of the parameters: \( A_1^2 = 1, A_2^2 = 1.2, B_1^2 = 0.8 \) and \( B_2^2 = 0.4 \). The Excel programme smoothed the calculated values in this and following figures and the dimensionless deflections \( v_1/v_0 \) and \( v_2/v_0 \) are depicted.

3.2 Low response of the system

As the speed parameter \( \alpha \) (14) possesses low values in practice (\( \alpha < 0.5 \) for railway bridges, while for highway structures even \( \alpha < 0.2 \)), we assume the limit
3.3 Effect of other input parameters

The effects of input parameters $A_{1}^{2}, A_{2}^{2}, B_{1}^{2}$ and $B_{2}^{2}$, Eqs (15) to (18), are depicted in Figures 3 to 6 for zero speed of the moving force, $\alpha = 0$. The first index belongs to the beam or string, respectively, while the second to various cases of input parameters as explained in Table 1.

All the values max $v$ are diminishing with increasing $A_{1}^{2}$ as can be seen in Fig. 3. The Fig. 4 shows the same tendency (valid for the parameter $A_{2}^{2}$). The parameter $B_{1}^{2}$ increases the value of max $v_{1}$, however, almost does not affect max $v_{2}$, Fig. 5. The parameter $B_{2}^{2}$ does not change the most important values of max $v_{1}$ but increases max $v_{2}$, Fig. 6.

The analysis of Figs 2 to 6 reads that it is really difficult to choice the optimum combination of five input parameters, $\alpha$ to $B_{2}^{2}$, which diminish the values of max $v_{1}$. Surely, the increasing speed of the moving force generally increases the dynamic response of the system for $\alpha \ll 1$ while the parameter $B_{2}^{2}$ produces only

![Figure 2: Effect of the maximum deflection of the beam max $v_{1}/v_{0}$ and that one of the string max $v_{2}/v_{0}$ on the speed parameter $\alpha$, Eq. (14).](image1)

![Figure 3: Max $v/v_{0}$ as a function of the parameter $A_{2}^{2}$ for $\alpha = 0$.](image2)
Table 1: Effect of input parameters for $\alpha = 0$.

<table>
<thead>
<tr>
<th>Parameters</th>
<th>$A_1^2$</th>
<th>$A_2^2$</th>
<th>$B_1^2$</th>
<th>$B_2^2$</th>
<th>Index of $v$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Effect of $A_1^2$</td>
<td>0.4 to 1</td>
<td>1</td>
<td>0.8</td>
<td>0.2</td>
<td>11 21</td>
</tr>
<tr>
<td></td>
<td></td>
<td>1.6</td>
<td>1.4</td>
<td>0.8</td>
<td>12 22</td>
</tr>
<tr>
<td>Effect of $A_2^2$</td>
<td>0.4</td>
<td>1 to 1.6</td>
<td>0.8</td>
<td>0.2</td>
<td>11 21</td>
</tr>
<tr>
<td></td>
<td></td>
<td>1</td>
<td>1.4</td>
<td>0.8</td>
<td>12 22</td>
</tr>
<tr>
<td>Effect of $B_1^2$</td>
<td>0.4</td>
<td>1</td>
<td>0.8 to 1.4</td>
<td>0.2</td>
<td>11 21</td>
</tr>
<tr>
<td></td>
<td></td>
<td>1</td>
<td>1.6</td>
<td>0.8</td>
<td>12 22</td>
</tr>
<tr>
<td>Effect of $B_2^2$</td>
<td>0.4</td>
<td>1</td>
<td>0.8</td>
<td>0.2 to 0.8</td>
<td>11 21</td>
</tr>
<tr>
<td></td>
<td></td>
<td>1</td>
<td>1.6</td>
<td>1.4</td>
<td>12 22</td>
</tr>
</tbody>
</table>

Figure 4: Max $v/v_0$ as a function of the parameter $A_2^2$ for $\alpha = 0$.

Figure 5: Max $v/v_0$ as a function of the parameter $B_2^2$ for $\alpha = 0$.

a little effect. Nevertheless, among the five parameters, a particular combination of input parameters could be found which produces low values of max $v_1$. It is valid especially for low speeds of the moving force, say $\alpha < 0.2$. However, the last mentioned case is important in practice of bridge engineering.
3.4 Input parameters for a real bridge

A real railway bridge, type PSKT, length 24 m, span \( l = 23 \) m, composed of two box girders, possesses the first natural frequency \( f_1 = 5.845 \) Hz and is prestressed by the force \( N = -29.32 \times 10^6 \) N. An elastic layer is supposed between the girders and tensile strings with the characteristic constant \( k \) (0; 5; 10; 20; 30 50 and 100 N/mm²) and with the mass per unit length \( \mu \) (0.5; 10^{-4}; 0.8379.10^{-4}; 1.10^{-4}; 2.10^{-4}; 3.10^{-4}; 5.10^{-4} and 10.10^{-4} \) Ns²/mm²). The force \( F \) is moving in the speed range \( \alpha [0; (0.1); 2] \).

Some of the input parameters are summarized in Table 2 and their effects are represented in Figures 7 to 9. Although Fig. 7 is a little analogous to Fig. 2, the first one provides a rising tendency of the maximum values max \( v_1 \) with increasing speed (for \( \alpha < 0.8 \)). The stiffness \( k \) of the elastic layer, represented in Fig. 8, does not affect too much the values of max \( v_1 \) and max \( v_2 \) (with the exception of very low values of \( k \) which characterize the absence of the elastic layer). The Fig. 9 shows a similar tendency where the mass \( \mu \) of the string diminishes both the values of max \( v \) only a little.

Differently from the previous Sections, the conclusions valid for a particular bridge show that the values of max \( v_1 \) can hardly be diminished by an elastic layer between the beam and string in this investigated case.

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**Figure 6**: Max \( v/v_0 \) as a function of the parameter \( B^2 \) for \( \alpha = 0 \).

**Figure 7**: Values of max \( v_1/v_0 \) and max \( v_2/v_0 \) as the function of the speed parameter \( \alpha \) (for \( k = 5 \) N/mm²).
Table 2: Input parameters for a real bridge and several values of $k$ and $\mu_2$.

<table>
<thead>
<tr>
<th>$k$ (N/mm$^2$)</th>
<th>0</th>
<th>5</th>
<th>50</th>
<th>100</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A_1^2$</td>
<td>0.9451</td>
<td>0.9918</td>
<td>0.9644</td>
<td>1.039</td>
</tr>
<tr>
<td>$A_2^2$</td>
<td>4.844</td>
<td>49.57</td>
<td>452.4</td>
<td>899.9</td>
</tr>
<tr>
<td>$B_1^2$</td>
<td>4.843</td>
<td>49.11</td>
<td>447.5</td>
<td>890.2</td>
</tr>
<tr>
<td>$B_2^2$</td>
<td>0</td>
<td>44.27</td>
<td>442.7</td>
<td>885.0</td>
</tr>
<tr>
<td>$\mu_2$ (Ns$^2$/mm$^2$)</td>
<td>$5.10^{-4}$</td>
<td>$2.10^{-4}$</td>
<td>$5.10^{-4}$</td>
<td>$10.10^{-4}$</td>
</tr>
<tr>
<td>$A_1^2$</td>
<td>0.9777</td>
<td>0.9733</td>
<td>0.9355</td>
<td>0.8766</td>
</tr>
<tr>
<td>$A_2^2$</td>
<td>157.5</td>
<td>40.10</td>
<td>16.66</td>
<td>8.895</td>
</tr>
<tr>
<td>$B_1^2$</td>
<td>156.5</td>
<td>39.12</td>
<td>15.65</td>
<td>7.826</td>
</tr>
<tr>
<td>$B_2^2$</td>
<td>148.4</td>
<td>37.09</td>
<td>14.84</td>
<td>7.418</td>
</tr>
</tbody>
</table>

Figure 8: Values of max $v_1/v_0$ and max $v_2/v_0$ as the function of the stiffness $k$ of the elastic layer (for $\alpha = 0.1$).

Figure 9: Values of max $v_1/v_0$ and max $v_2/v_0$ as the function of the mass $\mu_2$ per unit length of the string (for $\alpha = 0.1$).

4 Conclusions and discussion

A new complex system beam and string subjected to longitudinal forces and connected together with an elastic layer of Winkler type was introduced in the paper. The coupled system is subjected to a moving force. It represents a theoretical model of a prestressed bridge. Several thousand combinations of five input dimen-
Dimensionless parameters gave a survey of their effect on the basic form of vibration for time being.

It may be concluded that the dynamic effects roughly rise with the increasing dimensionless speed $\alpha$ up to its certain value (e.g. $\alpha < 1$) and then diminish. The other parameters show a complex effect on the maximum values of both the beam and string deflections that cannot be expressed in a simple way.

The main purpose of this paper was to find such a combination of input parameters that provide the maximum value of the beam deflection lower than one ($\max v_1/v_0 < 1$). It ought to be the most important conclusion for the practice of bridge engineering where usually $\alpha < 0.2$. Our attempts were partly successful.

Future studies must show the applicability of the complex system to the practice, especially how to build an elastic layer in the free space between the girders and pretensile strings. It is possible that the engineering practice do not find the parameters that diminish the basic form of the natural vibration, however, the diminishing of higher modes could produce small damages on bridge roadways and, thus, to contribute to engineering economy (maintenance savings) and safety.

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References


