Constrained minimization of incomplete quadratic functions and its applications

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Abstract

In an ordinary minimization problem, the quadratic function contains both quadratic and linear terms. Through minimization, a system of linear equations is yielded. The right-hand side vector of such a system is normally a known input. In a special type of elasticity problem in which coerced deformation of an elastic body is sought, the applied load becomes unknown. In other words, the quadratic function does not have a linear term. The objective function has two quadratic terms, and the minimization is conducted to satisfy some constraints. In another application where the amount of springback of a stamped metal is to be determined, the applied force giving the metal the least strain energy is sought. In this paper, the mathematical formulation as well as the approach to solve the problems is presented. Several numerical examples are presented.

Keywords: constrained minimization, quadratic, coerced deformation, springback, minimum strain energy, Lagrange multipliers.

1 Introduction

In some disciplines such as finding the displacements of a deformable elastic object, a solution $x_i$ is sought by finding the minimum of the following quadratic function.

$$U(x_i) = \frac{1}{2} a_{ij} x_i x_j + b_i x_i.$$  

(1)
While searching for such a solution, some conditions may have to be satisfied. For example, the following inequality constraint must hold true for a solution to exist.

\[ C(x_i) \leq 0, \]  \hspace{1cm} (2)

Mathematically, the above can be formulated as a constrained minimization problem [1], [2].

\[
\min_{x_i} \left( \frac{1}{2} a_{ij} x_i x_j + b_i x_i \right), \text{ subject to } C(x_i) \leq 0
\]  \hspace{1cm} (3)

Note that indicial summation convention is implied here.

This paper presents the constrained minimization of a couple of different types of quadratic function in which the second term in the above is either absent or replaced by another quadratic term. It further discloses the engineering applications of such a minimization problem, and presents a couple of application examples in the determination of springback of a formed sheet metal and coerced deformation of an elastic beam.

![Figure 1: A quadratic function with its quadratic, linear, and constant terms, and constraint function $C(x_i)$](image)

**2 Problem development**

**2.1 Constrained minimization of a complete quadratic function**

Let $x_i$ be the primary unknown variables, the complete quadratic function found in many engineering disciplines is given as follows.

\[ U(x_i) = \frac{1}{2} a_{ij} x_i x_j + b_i x_i + U_0, \]  \hspace{1cm} (4)
where $a_{ij}$ is a constant matrix, $b_i$ a constant vector, $U_0$ a constant scalar, and $\frac{1}{2}$ is added for convenience. The constrained minimization of the above function can be represented mathematically as below.

$$ \min_{x_i} U(x_i), $$

subject to

$$ C(x_i) \leq 0, $$

Graphically, eqn (4) can be represented in Figure 1 together with the constraint function $C(x_i)$. Note that the constant term $U_0$ bears no effect on the minimization, thus will be ignored in the following development.

### 2.2 Constrained minimization of an incomplete quadratic function

#### 2.2.1 Type I

There are cases that both the linear and the constant terms in function $U(x_i)$ to be minimized are absent. Let

$$ a_{ij} = \begin{bmatrix} k_{ij} & 0 \\ -2\delta_{ij} & 0 \end{bmatrix}, $$

and,

$$ x_i = \begin{bmatrix} u_i \\ f_i \end{bmatrix}, $$

where $k_{ij}$ represents the components of a stiffness matrix of an elastic body, $\delta_{ij}$ the Kronecker delta, $u_i$ the nodal displacements, and $f_i$ the nodal forces. Therefore, eqn (4) becomes, after setting $b_i = 0$,

$$ U(x_i) = \frac{1}{2} a_{ij} x_i x_j = \frac{1}{2} \begin{bmatrix} u_i \\ f_i \end{bmatrix}^T \begin{bmatrix} k_{ij} & 0 \\ -2\delta_{ij} & 0 \end{bmatrix} \begin{bmatrix} u_j \\ f_j \end{bmatrix}, $$

or,

$$ U(u_i, f_i) = \frac{1}{2} k_{ij} u_i u_j - f_i u_i. $$

Physically speaking, the above represents the discretized total potential energy of an elastic object [3] subject to an unknown nodal force vector. For example, the elastic beam shown in Figure 2 is to be forced to conform to the rigid barrier beneath it. The magnitude of the force, partially shown in Figure 2, is sought as well as the contact location on the rigid barrier for each material point on the beam. Note the similarity between eqns (4) and (10). This is
equivalent to saying that $b_i$ in eqn (4) is also part of the unknown primary variable and thus both the first and the second terms of $U(u_i, f_i)$ are quadratic. In this case, the constrained minimization problem becomes

$$\min_{u_i, f_i} \left( \frac{1}{2} k_{ij} u_i u_j - f_i u_i \right),$$

subject to

$$C(u_i) = 0.$$  \hspace{1cm} (11)

Here, the equality constraint is related to the distance between a node on the elastic beam and a surface point on the rigid barrier.

It is readily seen that the above minimization problem often leads to non-unique solutions, if any. Clearly one may apply a load to force the beam to the rigid barrier. Additional applied force in the same general direction will keep the beam in contact with the barrier. Therefore, another approach is sought to ensure a unique solution satisfying the constraint condition.

![Figure 2: An elastic beam to be forced to conform to a rigid barrier.](https://example.com/figure2.png)

$2.2.2$ **Type II**

Although eqn (10) has two quadratic terms, the two unknown variables in the second term, $f_i$ and $u_i$, are of different physical quantities having different units. Here, a second type of incomplete quadratic function with only one primary variable is considered in the constrained minimization. That is,

$$\min_{u_i, f_i} \left( \frac{1}{2} k_{ij} u_i u_j \right),$$

subject to

$$C(u_i) = 0.$$  \hspace{1cm} (13)

Note that without the constraint, eqn (14), the minimization would yield a trivial solution $u_i = 0$ as depicted in Figure 1. If the quadratic term in eqn (13) is interpreted as the discretized total strain energy in an elastic body, then the above minimization problem attempts to find a proper mean to force the elastic body to
deform to a specific way so that the least amount of strain energy stored in the elastic. This is similar to the problem depicted in Figure 2.

If one is to convert the above constrained minimization problem to one without constraint, the method of Lagrange’s multipliers may be employed, [1] and [2]. That is,

$$\min_{u, \lambda} \left( \frac{1}{2} k_{ij} u_i u_j - \lambda_i C_i \right), \quad (15)$$

where \( \lambda_i \) is the unknown Lagrange’s multiplier. It is assumed that there are a number nodes associated with the constraints. Analogy between eqn (11) and eqn (15) exists, although the term \( \lambda_i C_i \) is not quadratic at all. To appreciate this, a simple example shown in Figure 3 is used, where by a vertical force the free end of the cantilever beam is to contact the rigid barrier, which can be described by a parabolic equation. The constraint equation in this case is,

$$C \equiv v_1 - e(x_0 + u_1)^2 = 0, \quad (16)$$

where \( u_1 \) and \( v_1 \) are, respectively, the \( x \)- and \( y \)-displacements of the free end, and \( e \) is related to the eccentricity of the parabola. From eqn (15) the following matrix equation can be achieved for a one-element model.

\[
\begin{bmatrix}
\frac{AE}{L} & 0 & 0 & 2e(x_0 u_1 + u_1^2) \\
0 & \frac{12EI}{L^3} & -\frac{6EI}{L^2} & -1 \\
0 & -\frac{6EI}{L^2} & \frac{4EI}{L} & 0 \\
e(2x_0 + u_1) & -1 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
u_1 \\
v_1 \\
\theta_1 \\
\lambda_1
\end{bmatrix}
= \begin{bmatrix}
0 \\
0 \\
0 \\
- ex_0^2
\end{bmatrix}, \quad (17)
\]

where \( \theta_1 \) is the rotation of the beam at the free end. Here, a node of the beam element has 3 degrees of freedom. It is readily seen that the equation is highly nonlinear and nonsymmetrical. It is worth mentioning that one could apply some loads to cause the tip of the cantilever to contact the rigid barrier following either the path with the shortest distance, path I in Figure 3, or the path requiring the least effort, path II in Figure 3. The latter would be the solution to eqn (17).

Figure 3: An elastic beam whose tip is to be forced to contact a rigid barrier.
2.3 Engineering applications

In metal forming, springback is caused by the elastic recovery of the stress after metal has been released from the die and the punch. Excessive springback introduces deviations from the desired shape, thus causing quality concerns. It becomes vital to be able to determine the amount of springback accurately [4]. Recently, finite element analysis, for example Xia et al. [5] and Ogawa and Takizawa [6], has been widely used to simulate the metal forming process to accurately predict springback allowing its compensation to be taken into account in the design stage. Although a considerable amount of work has been done on the accuracy of springback prediction and optimization of springback reduction in design process, literature about how to accurately measure springback is very scarce. In stamping industry, the least distance from the point on stamped parts to die surface is usually used as the measure the amount of springback. In this paper, a constrained finite element method for assessing springback of various metals is proposed. In this method a formed part is forced back to the original location on the die surface with minimum stain energy as described in the constrained minimization eqn (13) and eqn (14).

3 Mathematical modelling and numerical examples

3.1 Coerced deformation of an elastic beam

Let the mathematical description of the surface of a simple rigid barrier be given as, referring to Figure 4,

\[ \bar{z} = f(\bar{x}, \bar{y}), \]  

(18)

where \((\bar{x}, \bar{y}, \bar{z})\) is an arbitrary point on the barrier surface. For simplicity, it is assumed that for each point on the \(x-y\) plane there is a unique point on the barrier surface. Therefore the displacements \((u_i, v_i, w_i)\) a point on the elastic body \((x_i, y_i, z_i)\) must acquire to reach the barrier surface are such that:

\[ z_i + w_i = f(x_i + u_i, y_i + v_i). \]  

(19)

Accordingly, the constraint for the point is defined as:

\[ C_i = z_i + w_i - f(x_i + u_i, y + v_j) = 0. \]  

(20)

In order for the elastic object seen in Figure 4 to deform to the rigid barrier, the following minimization problem with the unknown Lagrange multipliers \(\lambda_i\) must be solved.

\[ \min_{u_i, \lambda_i} \left( \frac{1}{2} k_{ij} u_i u_j - \lambda_i [z_i + w_i - f(x_i + u_i, y + v_j)] \right), \]  

(21)

where the stiffness matrix \(k_{ij}\) can be formed with ease [3]. Here, the deformation of the elastic object is confined in the elastic range.
Figure 4: Schematic view of an elastic object and a rigid barrier.

Figure 5: A multi-section elastic beam to be forced to the rigid barrier.

Figure 6: The finite element model of the elastic beam and the deflected beam.
In the first example, an elastic beam as prescribed in Figure 5 is to deflect to the rigid barrier leaving no gap between the beam and the rigid barrier. Note that the mathematical description of rigid barrier is given as

\[
(z - 1)^2 + (x + 0.2)^2 - 1 = 0, \quad -1.1 \leq x \leq -0.2 \\
\bar{z} = 0, \quad -0.2 < \bar{x} < 0.2, \quad (22)
\]

\[
(z - 1.5)^2 + (x - 0.2)^2 - 2.25 = 0, \quad 0.2 \leq \bar{x} \leq 1.3
\]

where the unit for the numerals is m. The elastic beam seen in Figure 5 also has two circular arcs and a straight part between the arcs. It is intentionally oriented in an arbitrary orientation (15° CCW as shown). This is to demonstrate in the minimization scheme that the rigid body motion does not provide any strain energy at all. For the elastic beam, the height and the width of the cross-section are \(h = 0.01\) m, \(b = 0.01\) m, respectively, and the Young’s modulus \(E = 200\) GPa. Furthermore, each node is assumed to have two translational degrees-of-freedom, and one rotational degree-of-freedom. Note that Newton-Raphson iteration method is used to solve the nonlinear problem.

Figure 6 depicts the finite element model of the undeformed elastic beam in Figure 5. For reference purpose, it is displayed with no rigid body rotation or translation. Twenty elements are used in the calculation. Figure 6 also shows that the displaced beam conforms perfectly to the rigid barrier.

### 3.2 Assessing springback of sheet metal under cylindrical bending

A thin sheet metal is clamped at the lower end and bent to the circular rigid die as shown in Figure 7(a). After bending, the punch (not shown) is removed for the sheet to release some of the stored strain energy. Note that this springback process is completely elastic [7], as a result the deformation for the bent sheet to return to the die is elastic. The dimensions for the bent sheet and the die are depicted in Figure 7(b). The die surface can be described mathematically as follows.

\[
\bar{x}^2 + \bar{z}^2 - 1 = 0, \quad \bar{z} > 0 \\
\bar{x} + 1 = 0, \quad \bar{z} \leq 0
\]

(23)

where the unit for the numerals is m. The flat part with 200 mm in length at the lower end is the area being clamped and has no deformation.

For cylindrical bending, the radius of the bent sheet, seen as 1,460.4 mm in Figure 7(b), can be predicted using the following formulas [8].

\[
\frac{1}{r} - \frac{1}{r'} = \frac{3\sigma_0}{tE'}, \quad E' = \frac{E}{(1 - \nu^2)}, \text{ and } \sigma_0 = 2Y/\sqrt{3}, \quad (24)
\]

where \(r\) denotes the radius of the die, \(r'\) the radius of the sheet after springback, \(\sigma_0\) the flow stress, \(t\) the thickness of the sheet metal, \(E\) the Young’s modulus, \(\nu\) the Poisson’s ratio, and \(Y\) the tensile yield stress. Note that plane strain has been assumed for the cylindrical bending. For the present example, the following material constants are used: \(E = 200\) GPa, \(\nu = 0.3\), \(Y = 200\) MPa. Thickness of the sheet is \(t = 0.01\) m.
Figure 7: (a) A sheet metal is bent to a rigid die and partially returns elastically, (b) the dimensions of the sheet metal after springback.

Figure 8: (a) Finite element model of the sheet and die in Figure 7, (b) enlarged view showing the deformed sheet metal in contact with the die surface.

Figure 8(a) displays the finite element model of the sheet metal together with the arc and short vertical segment representing the die surface. Note that four-noded isoparametric elements are used for the sheet metal. There is no explicit boundary constraints applied to the model. Figure 8(b) discloses the deformed sheet metal using the proposed algorithm. It is seen that the circular portion conforms to the circular die surface rather well. At the transition between the arc and the straight line, some nodes show deviation from the rigid die surface. This may be the result of the tolerance chosen.
4 Conclusions

A special class of quadratic functions having only quadratic terms subject to some constraint is presented. Engineering applications of minimizing such quadratic with constraints are shown. The constrained minimization scheme is nonlinear and quite complicated due to the nature of the complex constraint functions. The first example presented contains an elastic beam to be forced to contact a rigid barrier using the minimum amount of work. The proposed algorithm tackles this simple problem rather efficiently and accurate. In the second example, a plane strain model is used to assess the amount of springback a sheet metal has attained after a cylindrical bending. This method is shown more adequate the one practiced by industry where the least distance between a point on the stamped part and one on the die surface is used as a measure of the amount of springback. The algorithm calculates the deformed sheet metal using the constraint that all surface nodes are to return to the die surface. The result shown is also accurate except at the transition region where two mathematic functions are used to describe the die surface. Other simple constraint functions are used to test the algorithm with good agreement as reported here. Research is underway to tackle cases where the constraints cannot be presented by complete mathematic functions.

References