The effect of non-uniform gravity on mixed convection along a vertical surface

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Abstract

This paper considers the effect of non-uniform gravity on mixed convection along a vertical plate with an inverse-linear temperature distribution. The gravity varies with the distance from the leading edge of the isothermal plate. Two cases are taken into account: a) finite hot plate of length \( x_0 \), rotating at angular speed \( \omega \) in a radial plane about the line \( x = 0 \) (where \( x \) is the coordinate along the plate); b) infinite cold plate, rotating at angular speed \( \omega \) in a radial plane with its leading edge beginning at a distance \( x_0 \) from the axis of rotation. A pseudo-similarity transformation in the new variables \((\xi, \eta)\) is introduced in order to obtain a parabolic system of differential equations in non-similar form, which is solved by means of the Keller-box method. Using a non-uniform grid, concentrated towards the wall to enhance the resolution and accuracy, with the first step of \( \eta \) taken as 0.01, we used throughout the calculations a value \( \eta_{\text{max}} = 10 \) and a number of 241 points in the \( \eta \)-direction. The grid along the \( \xi \) axis was concentrated towards \( \xi = 0 \) and the number of points on this direction was 121. The level of precision imposed to the Keller-box scheme was 1e-10. The skin-friction coefficient and Nusselt number are obtained for various values of the Prandtl number and mixed convection parameter \( \lambda \). Both aiding and opposite effects are considered.
1 Introduction

Various researchers, Lienhard et al [1], Nath [2], Venkatachala and Nath (1979) have investigated the effect of non-uniform gravity due to the rotation of an isothermal plate. The first analysis of the non-uniform gravity fields caused by rotation of a plate was those of Lemlich and coworkers, see for instance Lemlich and Vardi [4]. Recently Kaloni and Qiao [5] have dealt with the variable gravity effects, in the nonlinear convection in a porous medium with inclined temperature gradient.

The gravity field may be created artificially in an orbital space station by rotation. Many other geothermal applications may involve centrifugal gravity fields.

On the other hand, the mixed convection along a vertical surface was studied in many studies, one of the most recent being that by Merkin and Pop [6]. In this paper similarity solutions are derived for mixed convection boundary-layer flow over a vertical semi-infinite flat plate in which the free stream velocity is uniform and the wall temperature is inversely proportional to the distance along the plate.

At our best knowledge, there are no studies in the open literature dealing with the effect of non-uniform gravity on the laminar mixed convection along a vertical plate. So, the objective of the present paper is to analyse the steady mixed convection heat transfer from a vertical plate placed in a viscous fluid and subjected to a general variation of gravity. We consider, as Merkin and Pop [6] did, that the free stream velocity is uniform and the wall temperature is inversely proportional to the distance along the plate.

2 Analysis

We consider the steady laminar mixed convection boundary-layer flow of a uniform free-stream $U_\infty$ along a flat plate immersed in a viscous incompressible fluid and subject to a non-uniform gravity field. In our study, there are considered two cases:

- finite hot plate of length $x_0$, rotating at angular speed $\omega$ in a radial plane about the line $x = 0$;
- infinite cold plate, rotating at angular speed $\omega$ in a radial plane with its leading edge beginning at a distance $x_0$ from the axis of rotation.

The governing equations are the continuity equation, the momentum equation and the energy equation, written as follows

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0,$$  \hspace{1cm} (1)

$$u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = g\beta (T - T_\infty) + \nu \frac{\partial^2 u}{\partial y^2},$$  \hspace{1cm} (2)
\[
\frac{\partial T}{\partial x} + \nu \frac{\partial T}{\partial y} = \kappa \frac{\partial^2 T}{\partial y^2},
\]

where \(x\) and \(y\) are co-ordinates, defined as usually, along the plate and normal to it plate, respectively. \(T\) is the temperature, \(T_a\) is the ambient temperature (taken as constant), \(g\) is the acceleration due to gravity, \(\nu\) and \(\kappa\) are the kinematic viscosity and thermal diffusivity, respectively and \(b\) is the coefficient of thermal expansion. The non-uniform gravity effect is described by the following equation

\[
g = \pm \omega^2 x_0 g_o \left(1 \pm \frac{x}{x_0}\right).
\]

The boundary conditions to be satisfied are

\[
\begin{align*}
u &= \rho = 0, & T &= T_a + T_0(x), & \text{on } y = 0, \\
u &\to U_a, & T &\to T_a, & \text{as } y \to \infty,
\end{align*}
\]

where \(T_0(x)\) is the temperature of the plate, assumed hereinafter to depend on \(x\) in the form

\[
T_0(x) = \Delta T \cdot \left(\frac{L}{x}\right).
\]

Here \(L\) is a length scale in the vertical direction and \(\Delta T\) is a scale for the applied temperature difference.

We look for similarity solutions in the form

\[
\psi = (\nu U_a x)^{1/2} f(\xi, \eta), \quad T - T_a = \Delta T \cdot \left(\frac{L}{x}\right) \Theta(\xi, \eta), \quad \eta = \left(\frac{U_a}{\nu x}\right)^{1/2} y,
\]

where \(\psi\) is the stream function, defined as usually

\[
u = \frac{\partial \psi}{\partial x}, \quad \nu = -\frac{\partial \psi}{\partial y}.
\]

We remark that in any case, \(\xi = x / x_0 \in [0,1]\). On the other hand, it is worth to point out, following Merkin and Pop [6], that the wall temperature distribution, as given in (6), appears to contain a mathematical singularity at the leading edge of the plate \((x = 0)\). This fact is just a consequence of how the similarity equations were set up. For other discussions related to the choice of this wall temperature distribution the reader is referred to [6].

Now equations (1-3) become

\[
\begin{align*}
f^{n+1} + \frac{1}{2} f^{n+2} \lambda \left(g / g_o\right) \Theta &= \xi \left(f^{n} \frac{\partial f}{\partial \xi} - f^{n} \frac{\partial f^{n}}{\partial \xi}\right), \\
\frac{1}{\sigma} \theta^{n+1} + \frac{1}{2} \theta^{n+2} \Theta &= \xi \left(f^{n} \frac{\partial \theta}{\partial \xi} - \theta^{n} \frac{\partial f}{\partial \xi}\right),
\end{align*}
\]

(9)
where \( \sigma = \kappa / \nu \) is the Prandtl number and \( \lambda = g \beta L \Delta T / U^2 \) is the mixed convection parameter. The case \( \lambda > 0 \) corresponds to a heated plate \( (\Delta T > 0, \text{i.e. aiding flow}) \), while \( \lambda < 0 \) represents a cooled plate \( (\Delta T < 0, \text{i.e. opposing flow}) \).

The boundary conditions become in transformed variables
\[
\begin{align*}
 f(\xi,0) &= 0, \quad f'(\xi,0) = 0, \quad \theta(\xi,0) = 1, \quad (11a) \\
 f'(\xi, \infty) &= 1, \quad \theta(\xi, \infty) = 0. \quad (11b)
\end{align*}
\]

To this end, we mention that equation (4) becomes
\[
g / g_0 = 1 \pm \xi. \quad (12)
\]

where the positive sign is for the cold rotating plate and the negative sign is for hot rotating plate.

The quantities of direct engineering interest are the skin-friction coefficient and Nusselt number, which are proportional to \( f''(\xi,0) \) and \( \theta'(\xi,0) \), respectively, see for instance Lienhard et al. [1].

3 Results

The problem (9)-(12) has been solved using a standard Keller-box implementation, see Keller and Cebeci [7], Cebeci and Bradshaw [8].

Figure 1: Variation of \( \theta'(\xi,0) \) for various values of the Prandtl number, when \( \lambda = 0 \).
Figure 2a: Variation of $\theta'(\xi,0)$ for $\sigma = 0.1$ and various values of $\lambda$, cold rotating plate.

Figure 2b: Variation of $\theta'(\xi,0)$ for $\sigma = 0.1$ and various values of $\lambda$, hot rotating plate.
Figure 2c: Variation of $f'(\zeta,0)$ for $\sigma = 0.1$ and various values of $\lambda$, cold rotating plate.

Figure 2d: Variation of $f''(\zeta,0)$ for $\sigma = 0.1$ and various values of $\lambda$, hot rotating plate.
Figure 3a: Variation of $\theta'(\xi,0)$ for $\sigma = 2.0$ and various values of $\lambda$, cold rotating plate.

Figure 3b: Variation of $\theta'(\xi,0)$ for $\sigma = 2.0$ and various values of $\lambda$, hot rotating plate.
Figure 3c: Variation of $f'''(\xi,0)$ for $\sigma = 2.0$ and various values of $\lambda$, cold rotating plate.

Figure 3d: Variation of $f'''(\xi,0)$ for $\sigma = 2.0$ and various values of $\lambda$, hot rotating plate.
A series of trials were first conducted in order to find out the appropriate value of \( \eta_{\text{max}} \), which denotes the maximum value of \( \eta \) delimiting the vertical direction of the grid. Using a non-uniform grid, concentrated towards the wall to enhance the resolution and accuracy, with the first step of \( \eta \) taken as 0.01, we used throughout the calculations a value \( \eta_{\text{max}} = 10 \) and a number of points 241 in the \( \eta \)-direction. The grid along the \( \xi \) axis was concentrated towards \( \xi = 0 \) and the number of points along this direction were 121. The level of precision imposed to the Keller-box scheme was \( 1 \times 10^{-10} \).

In Figure 1 there are shown the graphs of \( \theta'(\xi,0) \) vs. \( \xi \) when \( \lambda = 0 \) (forced convection), for various values of the Prandtl number, in the range (0.1 ... 10.0). Our results for \( \theta'(\xi,0) \) are in good agreement with those reported in [6]. On the other hand, it is worth to point out that for \( \xi = 0 \), with \( \lambda = 0 \), equation (9) reduces to the standard Blasius solution; our numerical scheme gives \( f^{'''}(\xi,0) = 0.33206 \), in full agreement with the classical results, see for instance [9].

Due to the space limitations, we restrict ourselves to present here only several graphs, corresponding to low and medium Prandtl numbers (\( \sigma = 0.1 \) and \( \sigma = 2.0 \)).

The case \( \sigma = 0.1 \) is displayed in Figure 2, where there are shown the variations of \( \theta'(\xi,0) \) and \( f^{'''}(\xi,0) \) along the plate for both cold and hot rotating plate cases. The range of \( \lambda \) is (0 ... 200), because once \( \lambda \) becomes negative, the convergence of the numerical method becomes worse. For example, at \( \lambda = -0.5 \), the numerical runs failed after approximately 6-7 values of \( \xi \), starting with \( \xi = 0 \). We remark that large values of \( \lambda \) mean free convection. Another feature is the slight variation along the plate of the Nusselt number, especially for the cold rotating plate case. The tendency is more pronounced for the skin friction, which increases/decreases along the plate in the cold/hot rotating plate case, as mixed convection parameter takes larger values.

In Figures 3a-d there are shown the graphs \( \theta'(\xi,0) \) and \( f^{'''}(\xi,0) \) as functions of \( \xi \), when the Prandtl number is \( \sigma = 2.0 \). Now the parameter \( \lambda \) has only negative values, due to similar reasons as above. The behaviour of \( f^{'''}(\xi,0) \) is the same as for is \( \sigma = 0.1 \), compare Fig. 2c with Fig. 3c and Fig. 2d with Fig. 3d. But the behaviour of \( \theta'(\xi,0) \) is now monotonically opposite, see Figures 2a-3a and Figures 2b-3b. Our numerical runs confirmed this change in the behaviour of \( \theta'(\xi,0) \) for other values of the Prandtl number greater than 0.1 (for example at \( \sigma = 0.7 \) and \( \sigma = 1.0 \)).

4 Conclusions

The solutions of the problem were obtained for various values of Prandtl number and different values of the mixed convection parameter. In the present paper there are presented some results in graphical form for two values of the Prandtl number, namely 0.1 and 2.0. More consistent results will be reported in a future paper. We point out here that as the Prandtl number is increased, the range of the
mixed convection parameter for which numerical solutions can be found becomes narrower.

Our results highlight also the critical role that the Prandtl number plays in the existence of the solutions, a conclusion supported also by the work of Merkin and Pop [6].

References


