Reliability of notched composite laminates using a combined experimental investigation and stochastic finite element analysis methodology

R. Ganesan & S.M. Venugopal
Concordia Centre for Composites (CONCOM), Department of Mechanical and Industrial Engineering, Concordia University, Canada

Abstract

The design of notched composite laminates is based on the averaged stress over certain characteristic distance from the notch edge and the strength of the corresponding un-notched laminate. In practice, the notch geometry is not perfect due to machining conditions and the mechanical properties of composite laminates. The imperfections in notch geometry have a stochastic distribution. On the other hand, tests on laminate coupons show that the un-notched strength also has a stochastic distribution. Accordingly, the characteristic length used in design has a stochastic distribution. Therefore, in order to achieve a design with a required reliability and safety, (i) the stress analysis of the notched laminate has to be conducted based on a stochastic approach, (ii) the strength distribution and its probabilistic parameters have to be determined through a number of tests, and (iii) the reliability analysis has to be conducted. The present work considers these objectives. A stochastic finite element analysis of notched symmetric cross-ply \([0/90]_4\), and angle-ply \([0_4/\pm 45]_2\), composite laminates is conducted. The hole shape is modeled using a hypotrochoid variation and further, the location of the hole centre is modeled as a Gaussian random variable. The resulting stress distribution and the averaged stress are determined and they are modeled using both Gaussian and Lognormal random variables. The distributions of the strength and the characteristic length of the laminates with symmetric cross-ply \([0/90]_4\), configuration are determined through testing 25 samples of notched and 25 samples of un-notched laminates. In a similar manner, tests on angle-ply laminate \([0_4/\pm 45]_2\), are conducted. The reliability indices for the two laminate
configurations are calculated by combining the stochastic finite element analysis and the test results.

1 Introduction

It is very hard to achieve a hole with a perfectly circular profile using the drilling operation on composite laminates in practical circumstances. Also, there is a possibility, that the driven hole is offset from the desired coordinates. For design purpose, a study of stress distribution in composite laminates that contain holes as mentioned above has to be conducted. Material properties such as the Young’s modulus, Poisson’s ratio and shear modulus of an orthotropic material display spatial variations over the laminate and also display sample-to-sample variations. Therefore stochastic finite element analysis has to be conducted in order to calculate the displacements and stresses in notched composite laminates.

As the strength of a perforated composite laminate is related to the in-plane elastic stresses within a region adjacent to the hole boundary, stress concentration factor is not the only parameter of consideration for the prediction of failure of notched laminates [1,2]. Accordingly, two stress parameters are of prime interest in the present work; maximum stress near the hole edge and equivalent stress calculated over the characteristic length.

In the present work, the terminologies such as controlled hole and uncontrolled hole laminate are used. Accordingly controlled hole laminate exhibits only the stochastic variation in material properties, while the uncontrolled hole laminate also exhibits the geometric variation of the hole, eccentricity of the hole and variation in the value of characteristic length.

2 Stochastic finite element analysis

Stochastic finite element analysis is carried out considering a two-dimensional eight-node isoparametric element. A typical eight-node element is shown in Figure 1. The shape functions employed are given below:

\[
N_i = \begin{cases} 
\frac{1}{4}((1 + \xi \eta)(1 + \eta \eta)(\xi^2 + \eta^2 - 1)) & : i = 1,2,3,4 \\
\frac{\xi^2}{2}(1 + \xi \eta)(1 - \eta^2) + \frac{\eta^2}{2}(1 + \eta \eta)(1 - \xi^2) & : i = 5,6,7,8
\end{cases}
\] (1)

![Figure 1: An eight-node element in the local co-ordinate system](image.png)
In the numerical evaluation of the integrals involved, three-point Gauss quadrature is used. Further details pertaining finite element formulation are available in Ref [3]. The stochastic field of Young's modulus in the fiber direction $E_1$ is given below and a similar procedure is applicable to other material parameters such as $E_2$, $G_{12}$, $\nu_{12}$, $\nu_{21}$, ply orientation angle and ply thickness.

\[ E_1 = E_1[1 + a(X)] ; \quad E[a(X)] = 0 \]  \hfill (2)

where $[a(X)]$ is a homogenous zero-mean stochastic field characterized by the auto-correlation function $R_{aa}(\xi)$ given by

\[ R_{aa}(\xi) = E[a(X)a(X + \xi)] \]  \hfill (3)

where $X$ represents the Cartesian-coordinate position vector $(x,y)$ of each Gauss point in the finite element mesh and $\xi$ is the separation vector between any two Gauss points. The set of sample values of $a(X)$ at discrete Gauss points is given by

\[ \{a\} = [L\{Z\}] \]  \hfill (4)

where $\{a\}$ is the vector $[a_1, a_2, ..., a_n]$, $n$ is the number of Gauss points in the finite element mesh, $\{Z\}$ is the one-dimensional vector of Gaussian random variables with zero mean and unit standard deviation, and $[L]$ is the lower triangular matrix obtained by Cholesky decomposition of the covariance matrix $[C_{aa}]$. The covariance matrix consists of coefficients $c_{ij}$ given by

\[ c_{ij} = Cov(a_i, a_j) = R_{aa}(\xi) \]  \hfill (5)

The Markov correlation model has been used to correlate the variations in the material properties from one point to another in the laminate and is given by

\[ R_{aa}(\xi) = \sigma_a^2 \exp\left[-\left(\frac{\xi}{d}\right)^2\right] \]  \hfill (6)

in which $\sigma_a$ is the standard deviation of the stochastic field $a(X)$ and further $d$ is a parameter called correlation length. Detailed description of the stochastic finite element method can be found in Ref. [4].

3 Experimental investigation

The value of characteristic length varies from one laminate configuration to another, but is assumed to have the same mean value for all loading conditions.
Experiments are conducted on all the selected laminate configurations. Accordingly, a symmetric cross-ply $[0/90]_h$, and an angle-ply $[0_2/\pm 45]_2$, composite laminate are chosen. A total of 25 samples of notched and 25 samples of un-notched laminates, having a width of 37.9 mm and gauge length of 180 mm are tested in tensile mode. Before conducting the test, specimens are checked for any defects, such as, delamination and voids. Ultimate strengths of notched and un-notched laminates, tested on a computerized MTS machine having a capacity of 10 tons are recorded online. Average stress failure criterion is used to calculate the value of characteristic length and is given by [2]:

$$\frac{1}{a_o} \int_{r=0}^{2\pi} \sigma'_r(x,0)dx = \sigma_o$$  (7)

where $\sigma'_r(x,0)$ represents the approximate solution of stress distribution along x-axis for infinite orthotropic plate, $a_o$ is the characteristic length and $\sigma_o$ is the ultimate strength of un-notched laminate. The final form of the equation for calculating the value of characteristic length is given by [2]:

$$\frac{\sigma'_n}{\sigma_o} = \frac{2(1-\nu)}{2(1-\nu) - \bar{\epsilon}^2 + (K^*_n - 3)(\bar{\epsilon}^n - \bar{\epsilon}^2)}$$

$$\bar{\epsilon}^n = \frac{R}{R + a_o}$$  (8)

where $R$ is the radius of the hole, $\sigma'_n$ is the ultimate strength of notched laminate, and $K^*_n$ is the stress concentration factor at the hole edge. Table 1 gives the mean ($\mu$) and standard deviation ($\sigma$) values of ultimate strength of notched and un-notched samples. The value of characteristic length $a_o$ for both laminate configurations is shown in the last column.

<table>
<thead>
<tr>
<th>Laminate Type</th>
<th>Notched</th>
<th>Un-notched</th>
<th>$a_o$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$[0/90]_{h}$</td>
<td>0.6201</td>
<td>0.0213</td>
<td>0.9484</td>
</tr>
<tr>
<td>$[0_2/\pm 45]_2$</td>
<td>0.8272</td>
<td>0.0479</td>
<td>0.9784</td>
</tr>
</tbody>
</table>

4 Description of hole geometry

A hypotrochoid variation in the hole shape is considered and the corresponding equation for variation is given by [5]:
where the term $g(\theta)$ is the factor by which the radius of a circle varies, $k = 7$ (a non-negative integer). Equation (9) is expanded in powers of $\varepsilon = 0.01$. $\theta$ is the angle at which a node is created on the circle while developing a finite element mesh. The maximum eccentricity of the hole from the hole center is of the order $(l/20)^2$ of an inch. Gaussian random values that describe the eccentricity of the hole are generated. As a result of change in hole location, the value of characteristic length ceases to remain constant. Assuming the variation of the characteristic length to follow a Gaussian distribution, a series of values are generated using a MATLAB® program.

5 Numerical results

The program validation is conducted for a $[0_4/\pm 45]_2s$ laminate considering a deterministic hole geometry and the complete laminate using the mean values of material properties of NCT-301 Graphite epoxy material that were determined experimentally [4]. In controlled hole laminates, variation in the material stiffness properties are taken into account, but the variation due to the geometric property is ignored. In the following example a $[0_4/\pm 45]_2s$ laminate that is 37.9 mm wide and has 151.6 mm gauge length is subjected to uniformly distributed load of 1.36 MN/m. Figure 1 depicts a comparison of MATLAB® program results with exact and ANSYS® solutions. Results are in excellent agreement with the reference results [2].

![Comparison of stress concentration factor](image)

Figure 2: Comparison of SCF of a $[0_4/\pm 45]_2s$ laminate obtained using MATLAB®, ANSYS® and exact solutions

5.1 Stochastic simulation of stress parameters

Stochastic simulation is carried out on complete uncontrolled hole laminate that is subjected to uniaxial and biaxial loads. The convergence of equivalent and
maximum stress parameters for $[0/90]_4$ and $[0_2/\pm 45]_2$ laminates under uniaxial load and biaxial load is shown in Figures 3 and 4 respectively.

Figure 3: Mean and standard deviation values of stress parameters of $[0_2/\pm 45]_2$ and $[0/90]_4$ laminates under uniaxial load.

From Figure 3-a it is observed that for $[0_2/\pm 45]_2$ laminate, fluctuation in the mean value of equivalent stress persists till 250 simulations and attains a constant value of 0.847 GPa at about 300 simulations. The mean maximum stress $\sigma_{\text{max}}$ shown in Figure 3-c attains a constant value after 150 simulations and the corresponding value is 1.385 GPa. It is to be noted that in addition to the stochastic variation of material properties, variations in the value of characteristic length and geometry of the hole also exist. Apparently these uncertain parameters contribute to a prolonged and non-uniform variation in the pattern. The trend remains the same when a comparison is made between the curve of the standard deviation of equivalent stress and that of the standard deviation of maximum stress as depicted in Figures 3-b and 3-d.

It is observed that the mean value of equivalent stress $\sigma_{\text{equ}}$ for $[0/90]_4$ laminate converges at about 250 simulations attaining a value of 0.808 GPa as shown in Figure 3-a. But observing Figure 3-c for mean maximum stress even after 250 simulations a constant value is not achieved and the corresponding standard deviation value also varies as depicted in Figure 3-d.
Figure 4: Mean and standard deviation values of stress parameters of \([0_2, \pm 45]_{2s}\) and \([0/90]_4s\) laminates under biaxial load.

It is seen from Figure 4-a to 4-d that for \([0_2, \pm 45]_{2s}\) laminate, all the plots attain a constant value at about 150 simulations itself. Although the trends of standard deviations of the parameters do not match, the variations of mean equivalent stress and mean maximum stress parameters do follow a similar trend. Considering the values of coefficient of variations of the mean equivalent stress and mean maximum stress one can see that the variation is slightly higher in the latter case, exceeding the former by 27.4%.

For \([0/90]_4s\) laminate mean value of equivalent stress attains a constant value of 0.6973 GPa at about 300 simulations and correspondingly its standard deviation is 0.0211 GPa. Similarly the mean value of maximum stress attains a constant value of 0.839 GPa and standard deviation of 0.0201 GPa at about 300 simulations. Considering the values of coefficients of variation of the equivalent stress and maximum stress parameters, one can see that there is an increase in the fluctuation in the former case by 25.5%.

6 Reliability analysis

In evaluating reliability, two parameters are considered; one representing the strength \((\sigma_s)\) and the other, the stress developed due to external loading
(\sigma_{\text{equ}} \text{or} \sigma_{\text{max}}). \text{ These parameters are used in the calculation of reliability of composite laminate.}

In the present work, Gaussian distribution and Lognormal distribution are used to model the probability density function (PDF) for the stress parameters and hence to calculate the reliability.

6.1 Reliability based on Gaussian distribution

The PDFs of the normal distribution of the stress parameter and the strength are given by

\[ f(\sigma_s) = \frac{1}{\sigma_s \sqrt{2\pi}} \exp \left[ -\frac{1}{2} \left( \frac{x - \mu_{\sigma_s}}{\sigma_{\sigma_s}} \right)^2 \right] \]

\[ f(\sigma_{\text{equ/\max}}) = \frac{1}{\sigma_{\sigma_{\text{equ/\max}}} \sqrt{2\pi}} \exp \left[ -\frac{1}{2} \left( \frac{x - \mu_{\sigma_{\text{equ/\max}}}}{\sigma_{\sigma_{\text{equ/\max}}}} \right)^2 \right] \]

(10)

Corresponding standardized variable \( z_R \) is given by

\[ z_R = \frac{\mu}{\sigma} = \frac{\mu_{\sigma_s} - \mu_{\sigma_{\text{equ/\max}}}}{\sqrt{\sigma_{\sigma_s}^2 + \sigma_{\sigma_{\text{equ/\max}}}^2}} \]

(11)

Entering the value \( z_R \) in Table A-10 of Ref. [6] area under the normal distribution curve corresponding to the combined population is found out. The reliability \( R \) is given by

\[ R = 1 - z_R \]

(12)

Equation (8) enables us to determine the standardized variable \( z_R \) corresponding to any desired reliability.

6.2 Reliability based on Lognormal distribution

In general the PDF of the lognormal distribution is given by

\[ f(x) = \frac{1}{x \sigma_y \sqrt{2\pi}} \exp \left[ -\frac{1}{2} \left( \frac{\ln x - \mu_y}{\sigma_y} \right)^2 \right] \]

(13)

where distribution of \( y \) is a subsidiary distribution to \( x \sim LN(\mu_y, \sigma_y) \) and
\begin{equation}
\mu_\sigma = \ln \mu_s - \ln \sqrt{1 + C_s^2} = \ln \mu_s - \frac{C_s^2}{2}
\end{equation}

\begin{equation}
\sigma_\sigma = \sqrt{\ln(1 + C_s^2)} = C_s
\end{equation}

In the present case, subscript x assumes the stress and the strength parameters as shown in section 6.1. These equations make it possible to use Table A-10 in Ref. [6] in calculating the reliability.

### 6.3 Reliability calculation

Reliability indices are found out using equations (10-15). The values thus calculated for uniaxial and biaxial loads for \([0_\circ / \pm 45_\circ]_s\) and \([0/90]_t\) laminates are listed in Tables 2 and 3. Table 2 contains the reliability indices calculated using Gaussian distribution and Table 3 gives the results obtained using Lognormal distribution.

<table>
<thead>
<tr>
<th>Reliability parameters</th>
<th>Reliability for ([0_\circ / \pm 45_\circ]_s)</th>
<th>Reliability for ([0/90]_t)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\sigma_o)</td>
<td>Uniaxial load</td>
<td>Uniaxial load</td>
</tr>
<tr>
<td>(\sigma_{\text{max}})</td>
<td>1.000</td>
<td>0.999</td>
</tr>
<tr>
<td>(\sigma_{\text{equ}})</td>
<td>0.9726</td>
<td>0.9998</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Reliability parameters</th>
<th>Reliability for ([0_\circ / \pm 45_\circ]_s)</th>
<th>Reliability for ([0/90]_t)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\sigma_o)</td>
<td>Uniaxial load</td>
<td>Uniaxial load</td>
</tr>
<tr>
<td>(\sigma_{\text{max}})</td>
<td>1.000</td>
<td>0.999</td>
</tr>
<tr>
<td>(\sigma_{\text{equ}})</td>
<td>0.9699</td>
<td>0.9998</td>
</tr>
</tbody>
</table>

### 7 Conclusions

In the present paper, a study on the convergence of the probabilistic parameters of stress distribution with the number of simulations has been carried out on NCT-301 graphite epoxy laminates. Two laminate configurations viz., \([0_\circ / \pm 45_\circ]_s\) and \([0/90]_t\) under uniaxial and biaxial loading conditions are considered. Micro structural study revealed no delamination in the sample lot. From experimental results, as tabulated in Table 1 it is seen that the \([0_\circ / \pm 45_\circ]_s\) laminate configuration can bear a higher load in uniaxial tensile mode when
Compared with \([0/90]_{4s}\) laminate. But the number of simulations required for the convergence of stress values in uniaxial mode is more for the \([0_\pm 45]_{4s}\) laminate as opposed to \([0/90]_{4s}\) laminate and vice versa in biaxial mode. A reliability study is conducted with a factor of safety of 1.2 on the ultimate load. It can be observed from Gaussian and Lognormal distribution methods that both the reliability values corresponding to the sets \((\sigma_o, \sigma_{\text{max}})\) and \((\sigma_o, \sigma_{\text{equ}})\) for \([0_\pm 45]_{4s}\) laminate in uniaxial load condition exceed 97\%. For the biaxial load case, as seen from Tables 2 and 3, probability of failure of laminate corresponding to the maximum stress \(\sigma_{\text{max}}\) is higher than that corresponding to the equivalent stress \(\sigma_{\text{equ}}\). Thus in designing the \([0_\pm 45]_{4s}\) laminate, precautions have to be taken to improve the reliability corresponding to \(\sigma_{\text{max}}\). For \([0/90]_{4s}\) laminate the trend is opposite to the previous case in that the reliability is high in the biaxial mode and a reduction in the reliability is present in uniaxial mode (88.1\% reliability). For the same amount of load the reliability corresponding to the equivalent stress parameter \(\sigma_{\text{equ}}\) for the uniaxial load has decreased when compared with the value corresponding to the maximum stress \(\sigma_{\text{max}}\). Thus, measures must be taken to improve the reliability while designing a \([0/90]_{4s}\) laminate. An increase in reliability for \([0_\pm 45]_{4s}\) laminate in biaxial load mode and for \([0/90]_{4s}\) laminate in the uniaxial load mode can be achieved by increasing the factor of safety, and the achieved reliability for a factor of safety of 1.75 is 0.9984 and 0.9999 respectively.

References


