Filtering of experimental data at arbitrarily located points of planes and surfaces

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Abstract

The paper presents generalization of the method previously developed for regular meshes for filtering of the experimental or numerical data [1, 2]. The data obtained from measurements or numerical analysis of various technical problems always contain some errors caused by inaccuracies of measuring techniques and instruments and also due to some imperfections of the numerical methods. The generalization presented here makes it possible to use the filtering technique in the cases when the data is collected at arbitrary points of surfaces, or time, and is not necessarily located on regular rectangular meshes.

The obtained data is usually collected at arbitrarily located points at the surfaces or inside the analyzed bodies. The data can be arbitrary in nature. However, it always has to satisfy certain constraints resulting from the nature of the analyzed problem. The paper presents the algorithm which makes possible the elimination of errors or noise using a very fast and simple operation.

1 Introduction

The results of the measurements or the results of the numerical analysis (for example from the Finite Element Method) always contain some errors caused by inaccuracies of measuring techniques and instruments and also errors
It is assumed that the data are collected at arbitrarily located points of the surface or inside the analyzed bodies specified only by their coordinates. This data, however, has to satisfy certain constraints resulting from the nature of the analyzed problem. For example, the stresses in the body have to satisfy the conditions of equilibrium and also equations of continuity of the deformations. These equations can be used as the constraint equations that have to be imposed on the measured or calculated data. The constraint equations can result from any conditions imposed on the analyzed body, they can represent the conditions of symmetry, additional boundary condition. Even some technical or optimization requirements can be used as constraint equations.

The approach presented here can be utilized in analyzing of the data in thermal problems, dynamic problems, in identification techniques and Finite Element applications. The most important advantage of the matrix filter algorithm is that the correction of the data is a very fast operation and can be performed in real time while the measurements are performed.

The results obtained from the finite element (FEM) analysis are most often obtained at arbitrary nodal points. Due to the fact that the FEM solutions are usually based on displacement method the boundary conditions represented in terms of stresses are often not exactly satisfied. The filtering algorithm [1] offers a simple and effective tool to quickly correct these errors. It is proved that this approach can give the correction also in the case of the wrong order of singularities of the stresses at the singular points or at the edges where the boundary conditions are specified in terms of stresses. For example, the corrections can be done at the free edges of the body where FEM calculation often gives some unrealistic stresses. Examples of the application of the method to the plate and shell problems are here presented.

2 Formulation of the matrix filter algorithm

To obtain the corrected data \( u_i \), the vector of the raw data \( u_i^* \) is multiplied by the filter matrix \( M \) of the examined system.

\[
  u_i = M u_i^*,
\]

where: \( M = I - H^T (HH^T)^{-1} H \) and \( H u_i = 0 \). (1)

The matrix \( M \) is obtained from the constraint equations for the analyzed system and must be created only once. In the above equations \( I \) is the unit matrix, the matrix \( H \) represents the constraint equations in terms of finite difference operators. The correction of the data requires only the calculation of the product of the vector \( u_i^* \) and the matrix \( M \), which is a simple and fast operation. The calculation of the matrix \( M \) is more time consuming. It requires the representation of the system by means of the matrix \( H \) and calculation of its inversion. In the present paper it is shown
how the matrix \( M \) can be calculated on irregular points by introducing the generalized differential operators. These operators are based on the development of a function in Taylor series.

3 Plane problem

The problem of filtering is reduced to the creation of the matrix \( H \) and matrix \( M \) according to the equations (1). It should be emphasized here that this matrix does not consist of the complete set of equations for the discussed problem. It is built using only the information available for the analyzed system.

Two different plane differential operators are discussed in particular. The schemes of the six – point operator and the nine – point operator are presented in Fig. 1.

![Figure 1. The schemes of the six and nine point operators (stars).](image)

### 3.1 Six – point star operator

Using Taylor series representation of the function \( f \) for the six – point operator, the derivatives of the function \( f \) at the central point can be calculated from the equation:

\[
f = Q D f,
\]

where \( f \) is a vector of the values of the function at the points of the star,

\[
f = \begin{bmatrix} f_1 - f_0 \\ f_2 - f_0 \\ \vdots \\ f_5 - f_0 \end{bmatrix}
\]

\( Q \) is the matrix of the coefficients,
3.2 Stars of more then six points

When the nine-point operator is used, the system of equations becomes overdetermined. In this case it can be solved by means of the least square method [3]. We can approximate the function \( f \) requesting that the error \( R \) between the function and their Taylor series expansion is minimum. This error can be formulated as:

\[
B = \sum_{j=1}^{8} \left[ \left( f_j - f_0 + \sum_{k=1}^{2} \frac{1}{k!} \left( \sum_{i=1}^{2} \Delta x_{ij} \frac{\partial}{\partial x_i} \right)^k f_0 \right) \ast \frac{1}{\rho_j^3} \right]^2
\]

(6)

where
- \( j \) - number of the star,
- \( k \) - coefficient of the Taylor series,
- \( \Delta x_{ij} \) - local \( i \)-coordinate of the point \( j \),
- \( f_0 \) - value of the function in the central point,
- \( \rho_j = \sqrt{\sum_{i=1}^{2} \Delta x_{ij}^2} \) - distance of point \( j \) from the central point.

Minimizing the error with respect the unknown values of the derivatives in the central point the following set of equations can be obtained

\[
C \ast f = C \ast Q \ast Df
\]

(7)

where \( C \) is the matrix of the coefficients of the local coordinates and weights.
The above system of equations can be solved for each star providing the required derivatives of the function $f$. Then the matrix $H$ of the constraints is built.

4 Examples

To illustrate the presented approach the filtering algorithm was used to eliminate errors in measurements of the stresses in a two-dimensional problem of the theory of elasticity. In this case the governing equations which can be used as constraints conditions are:

$$\Delta (\sigma_{xx} + \sigma_{yy}) = 0,$$

and

$$\frac{\partial^2 \sigma_{xx}}{\partial x^2} - \frac{\partial^2 \sigma_{yy}}{\partial y^2} = 0. \quad (9)$$

4.1 Example 1

Figs. 2a present the raw data in a plate where the stresses were introduced by means of the edge loads. Figs. 2b present the results of filtering of the errors. The stresses were measured at the arbitrarily located points (the dots in the figure). Three local errors were introduced to the exact stresses created using the stress function [4].
4.2 Example 2

The second example presents the results of the filtering for a plate loaded by a concentrated normal force applied at the free edge of a semi-infinite plate.
Fig. 3a presents the points where the data were obtained and corrected. Fig. 3b depicts the exact results obtained from the solution of the equations of the theory of elasticity. Fig. 3c presents the results with a large initial error, Fig. 3d presents the corrected results and Fig. 3e and 3f the contour lines for the stress before and after filtering. Maximum initial relative error is in this example equal to 2.08. Maximum corrected relative error is 0.14. (Fig. 3h). The relative error is defined as the difference between the actual value and the exact value at a given point divided by the exact value at the same point. Fig. 3g presents the exact results. The 6-point operators were used here.

![Fig. 3g](image1.png) ![Fig. 3h](image2.png)

### 4.3 Example 3

The results of filtering of the stresses around a hole in a strip under tension are presented in Figs. 4a - 4h. The points are distributed arbitrarily. (Fig. 4a)

![Fig. 4a](image3.png) ![Fig. 4b](image4.png)

Fig. 4b presents the stresses around the hole affected by errors in the vicinity of the hole. Fig. 4c and 4d present the results after filtering and the exact results. 9-point operators were used here.
4.4 Example 4

In this example the distribution of the temperature in a spherical shell is presented. The shell is heated at the lower edge. The initial data were calculated by FEM and some error was introduced. The governing equation for this problem is $\Delta T = 0$. 

Figs. 5a and 5b present the shell and the points where the data were introduced.
Fig 5b

Fig 5c

Fig 5d

Fig 5e

Fig 5f

Fig 5g
Figures 5c and 5d present the raw data with large initial error. Figures 5e and 5f present the corrected results. Figure 5g shows the contour lines for the exact results.

5 Conclusions

It has been proved that the Filter Matrix method can be successfully used for filtering the data at arbitrary points of planes and surfaces. The method works well. However, it can be noticed, that the largest errors appear at the edges of the areas where the data was filtered. To avoid these errors we can disregard the filtered data at the edges or expand the filtered areas. We can also, if possible, introduce the boundary conditions at the edges as the constraint equations. The edge errors can be completely eliminated when at least a row of points is left between the edge and the analyzed area. The application of 9-point operators usually greatly increases the accuracy of the filtering. The larger number of points always provides better results.

References: