Dynamic behavior of a damped spring-mass system subjected to regenerative and modulated forces

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Abstract

This paper presents an analysis of the dynamic characteristics of a single degree-of-freedom spring-mass-dashpot system subjected to regenerative and modulated forces. This model is relevant to the basic dynamics of machine tools and industrial saws that may suffer from instability problems. It is found that the coupling effect of the regenerative and modulated forces results in different magnification factors for the different frequency components of the exciting force. The highest vibration level appears within a narrow frequency band corresponding to the sum of the damped natural frequency and the modulating frequency. The effects of the regenerative force and damping on the unstable regions are examined.

1 Introduction

Many machining processes suffer from problems such as, chatter in machine tools, washboarding in industrial saws and waviness in slitters. These problems are caused by severe relative vibrations of the cutter with respect to the work-piece. Vibration instabilities in these systems have been identified as the primary cause of such problems (Tobias [1], Chiriacescu [2], Okai et alia [3], Tian and Hutton [4]). According to the undamped stability analysis conducted by Tian and Hutton [5], the primary unstable regions cover a wide frequency range, especially in a system with high modal densities. However, experimental results indicate that the actual instability regions are more confined than those predicted by the undamped analysis. The experimental results also show that when the excitation frequency is slightly higher than the natural frequency the system will be unstable (Okai et alia [3], Tian and Hutton [5]). However the system response
is found to be stable in other regions where the undamped analysis predicts instability.

Time-dependent disturbing forces, such as edge impacts in milling and tooth impacts in sawing, can also cause severe vibrations due to the resonance of these systems (Tobias [6]). Usually, the operating speed of a system is such that the excitation frequency is sufficiently far from a resonance frequency. In this case, the forced vibration response is often negligible.

The dynamic characteristics of a single-degree of freedom damped spring-mass system subjected to both self-induced excitation, due to a regenerative force, and a time-dependent modulated excitation are discussed in this paper. Coupling effects of the self-excited vibration and the forced vibration demonstrate that maximum vibrations can exist in the model when the primary excitation frequency is different from the natural frequency. The influence of the model parameters on the system behavior is discussed.

2 Modeling of the system

A single-degree of freedom spring-mass-dashpot system as shown in Figure 1 has mass $m$, damping coefficient $c$, spring constant $k$ and is subjected to a regenerative force

$$F_{reg} = -k_1 [x(t) - x(t - T)]$$

(1)

where $k_1$ is the regenerative force coefficient, $x(t)$ is the displacement and $T$ is the period of a disturbing force. This type of regenerative force occurs in both metal and wood machining processes (Tobias [6], Tian and Hutton [4]). During milling or sawing, the cutter edges or teeth experience periodic impacts at the beginning of the cut. These impacts take place at the tooth passing frequency. The impact forces depend on the geometry of the teeth and their relative

![Figure 1: A single-degree spring-mass-dashpot system subjected to a regenerative force and a dynamic force](image)

$(m = 1 \text{ kg}, k = 500 \text{ kN/m}, k_1 = 10 \text{ kN/m}, \zeta = 0.002)$
velocities to the work-piece. In this work, it will be assumed that the primary force has frequency $\omega_t$ and is modulated by a frequency $\omega_r$. Thus, the exciting force in this model is assumed to have the form

$$f(t) = A_0(1 + a \cos \omega_r t) \cos \omega_t t$$

(2)

$$= A_0 \cos \omega_r t + A_0 a[\cos \omega_r(1 - p)t + \cos \omega_r(1 + p)t]$$

where $A_0$ is the amplitude of the force function, $a$ the amplitude of modulation function, $\omega_r$ the modulating frequency and $\omega_t$, the excitation frequency (corresponding to a tooth passing frequency for example). $p$ is the ratio of the modulating frequency to the excitation frequency, i.e., $p = \omega_r / \omega_t$.

This system is governed by the equation of motion

$$m \ddot{x}(t) + c \dot{x}(t) + kx(t) + k_1[x(t) - x(t - T)] = f(t)$$

(3)

or

$$\ddot{x}(t) + 2\zeta \omega_n \dot{x}(t) + \omega_n^2 x(t) + \omega_c^2[x(t) - x(t - T)] = f'_1(t)$$

(4)

where

$$\omega_n^2 = \frac{k}{m}, \quad \omega_c^2 = \frac{k_1}{m}, \quad \zeta = \frac{c}{2m\omega_n}, \quad T = \frac{2\pi}{\omega_t}, \quad f'_1(t) = \frac{f(t)}{m}$$

The homogeneous equation is

$$\ddot{x}(t) + 2\zeta \omega_n \dot{x}(t) + \omega_n^2 x(t) + \omega_c^2[x(t) - x(t - T)] = 0$$

(6)

Assume that the displacement be expressed by $x(t) = X_0 e^{\lambda t}$. The characteristic equation is then given by

$$\lambda^2 + 2\zeta \omega_n \lambda + \omega_n^2 + \omega_c^2(1 - e^{-\lambda T}) = 0$$

(7)

The complex eigenvalues are related to the damping ratio $\zeta$, the regenerative force constant $k_1$ and the period $T$ or the excitation frequency $\omega_t$. The excitation frequencies at which the real parts of its eigenvalues are positive can be used to define the unstable regions of this system.

The exciting force $f(t)$ in eqn (3) can be decomposed into three components at frequencies $\omega_r(l-p)$, $\omega_t$, and $\omega_r(l+p)$, respectively. The homogeneous solution of this equation will be attenuated due to the damping as time advances. The particular solution of this linear system is the sum of the solutions to three exciting forces. Each of the terms can be expressed in the general form,

$$f(t) = F_0 \cos \omega_r t = F_0 \cos 2\pi Ft$$

(8)
where
\[
\omega = \omega, \quad \omega_i(1 - p) \quad \text{or} \quad \omega_i(1 + p) \quad (9)
\]
\[
F = F, \quad F_i(1 - p) \quad \text{or} \quad F_i(1 + p) \quad (10)
\]
\[
F_0 = A_0 \quad \text{or} \quad A_0 a \quad (11)
\]

Thus the steady state response to this general excitation can be expressed by
\[
x(t) = A \cos \omega t + B \sin \omega t \quad (12)
\]
and
\[
x(t - T) = A \cos \omega (t - T) + B \sin \omega (t - T)
\]
\[
= (A \cos \omega T - B \sin \omega T) \cos \omega t
\]
\[
+ (A \sin \omega T + B \cos \omega T) \sin \omega t
\]

where \(A\) and \(B\) are two constants given in the form
\[
A = \frac{[\omega_n^2 - \omega^2 + \omega_c^2(1 - \cos \alpha)]F_0 / m}{[\omega_n^2 - \omega^2 + \omega_c^2(1 - \cos \alpha)]^2 + (2\zeta \omega_n \omega + \omega_c^2 \sin \alpha)^2} \quad (14a)
\]
\[
B = \frac{(2\zeta \omega_n \omega + \omega_c^2 \sin \alpha)F_0 / m}{[\omega_n^2 - \omega^2 + \omega_c^2(1 - \cos \alpha)]^2 + (2\zeta \omega_n \omega + \omega_c^2 \sin \alpha)^2} \quad (14b)
\]

where
\[
\alpha = 0, \quad -2\pi p \quad \text{or} \quad +2\pi p \quad (15)
\]
and \(\omega = \omega, \quad \omega_i(1 - p) \quad \text{or} \quad \omega_i(1 + p)\), respectively.

The magnitude of response to the exciting force is given by
\[
|x(t)| = \sqrt{A^2 + B^2} = X(\omega)F_0/k \quad (16)
\]

where the magnification factor
\[
X(\omega) = \frac{1}{\{1 - r^2 + r_c^2(1 - \cos \alpha)]^2 + (2\zeta r + r_c^2 \sin \alpha)^2\}^{\frac{1}{2}}} \quad (17)
\]
\[
r = \frac{\omega}{\omega_n}, \quad r_c = \frac{\omega_c}{\omega_n} \quad (18)
\]

The response to the excitation (2) can be expressed in the form
\[
x(t) = \frac{A_0}{k} \{X(\omega_i)\cos(\omega_i t + \phi_0) + a X(\omega_i(1 - p))\cos(\omega_i(1 - p)t + \phi_1) + a X(\omega_i(1 + p))\cos(\omega_i(1 + p)t + \phi_2)\} \quad (19)
\]
where $\phi_i$, $i = 0, 1, 2$ are the phases. It is seen that the vibration amplitude is directly proportional to the force amplitude $A_0$, the modulation factor $a$ and the magnification factors $X(\omega)$.

### 3 Instability regions

Figure 2 (a) shows that the natural frequencies of the system with different damping ratios ($\zeta = 0$, 0.002, 0.006 and 0.012) are almost same. The unstable regions of the system are defined as the frequency range in which the real parts of the eigenvalues are positive. It is seen from Figure 2 (b), the undamped system ($\zeta = 0$) has the widest primary instability region, i.e., $1/T = (F_n, 2F_n)$. Other primary unstable regions of the damped system shrink with an increase of damping. Furthermore, there is a critical damping ratio corresponding to the lightest damping required to make the system stable.

![Figure 2](image)

**Figure 2:** Natural frequency (a) and real part (b) of the system with different damping ratios ($k_f = 10$ kN/m).

The instability of the system is also affected by the regenerative force. A greater regenerative force increases the real part of eigenvalue of a undamped system (Figure 3(a)). It does not change the unstable region (Bolotin [1], Tian and Hutton [4]). In a damped system, however, this force can not only increase the real part of its eigenvalue, but enlarge the unstable regions (Figure 3(b)). In addition, it is noted that there is the critical regenerative force coefficient that is the smallest value of the coefficient for which the system is stable for the given level of damping. The effects of the damping and the regenerative force on the instability of this system are coupled with each other.
4 Magnitude of response

4.1 Effect of excitation frequency

Figure 4 shows the magnification factors for response around the natural frequency \( F_n = 112.5 \) Hz of the system to three excitation components at the frequencies \( F_1 (1-p) \), \( F_1 \), and \( F_1 (1+p) \). From eqn (14) it can be seen that the peak response of the system is effected by the characteristics of the regenerative force model. Without the amplitude modulation, the peak magnitude occurs at the damped resonance frequency \( F_d \). As a result of the influence of the regenerative force, the peak magnification factor due to the excitation at the frequency \( F_1 (1-p) \) is greater than those at the damped resonance frequency \( F_d \) and that due to the excitation at the frequency \( F_1 (1+p) \). In fact, it can be seen from eqns (15) and (17) that at the excitation frequency \( F_1 (1+p) \), i.e., \( \alpha > 0 \), the regenerative force within a certain range is equivalent to a positive damping in this system. On the other hand, if \( \alpha < 0 \), the regenerative force within a certain range is equivalent to a negative damping in this system.
In addition, if the excitation frequency $F_I$ is higher than the natural frequency (as shown in Figures 2 and 3), the real part of eigenvalue may be positive and the system may therefore experience instability.

Figure 5 (a) shows how the magnification factors vary with the excitation frequency for different frequency ratios. If $p \leq 0.01$, the peak magnification is small. If $p \geq 0.06$, the peak response is also small. When $p = 0.03$, the peak magnification factor achieves a maximum value. The frequency ratio at which this peak occurs is related to the damping ratio and the amplitude of regenerative force coefficient.

Figure 5: Variation of magnification factors with frequency ratio
($\zeta = 0.002, k_I = 10\, \text{kN/m}$).
4.2 Effect of damping

Figure 7 shows how the magnification factors vary with the damping ratio. The peak magnifications decrease with increasing damping ratio when \( p = 0 \) or the damping ratio is greater than a critical value, in this case, \( \zeta = 0.004 \). For the response to the excitation frequency \( F_i (1-p) \), when \( \alpha < 0 \) in eqn (17), the regenerative force is equivalent to negative damping on this system. Then the effective damping ratio becomes smaller with an increase of damping and the peak magnification increases.

4.3 Effect of regenerative force

The effect of the regenerative force on the peak magnification factors can be clearly seen in Figure 8. The response component at the frequency \( F_i (1-p) \) is greater than the components at frequencies \( F_i \) and \( F_i (1+p) \) if the force coefficient \( k_f \leq 30 \text{ kN/m} \). For the excitation component at frequency \( F_s \), the regenerative force does not affect the magnification factor. For the component of frequency \( F_i (1+p) \), the regenerative force always provides an equivalent positive damping and an extra stiffness for this system. For the excitation of frequency \( F_i (1-p) \), on the other hand, the regenerative force can cause negative damping when the force is small and positive damping when the force coefficient is greater than a critical value.
Figure 7: Variation of magnitude with damping ratio ($\rho = 0.02$, $k_l = 30$ kN/m).

Figure 8: Variation of Maximum magnitude with force coefficient ($\rho = 0.02$, $\zeta = 0.002$).

5 Conclusions

An analysis of the forced vibration response and stability characteristics of a single degree-of-freedom spring-mass-dashpot system subjected to regenerative and modulated forces has been presented. The results demonstrate the relationships between the response, the damping ratio, the regenerative force, and the modulated excitation. The results show:

Increasing the damping in this system reduces the unstable regions and the real part of the eigenvalues decrease. The largest unstable regions appear when no damping exists in this system. There is a damping ratio above which the unstable regions of the system disappear. The unstable regions can be extended
by an increase of the regenerative force coefficient. If this coefficient is small enough, the damped system is always stable.

Among the three forced vibration components, the component that gives rise to the maximum magnification factor appears at the frequency that is the sum of the damped natural frequency and the modulating frequency.

Although the primary instability region of this system extends from the natural frequency to twice this frequency, the highest vibration level appears within a narrow frequency band with center frequency $F_i (1-p)$. This is not only because the damping contracts the unstable region, but also because the modulated exciting force shifts the peak magnification to the higher excitation frequency.

If the excitation frequency is higher than the natural frequency, the peak response is at the frequency $F_i (1-p)$. The magnification factor, which is smaller than that at the resonance frequency, depends on the magnitude and frequency of the excitation and the behavior of the transfer function in this frequency range. It is possible that the forced vibration dominates the vibration response if the system suffers from a serious amplitude modulation.

The effect of damping on the magnification of this system is closely coupled with the effect of the regenerative force. There are combinations of the damping ratios and the regenerative forces that make the system give rise to maximum magnification factors. On the other hand, other combinations exist that minimize the magnification factor.

References