Two-dimensional transient shear wave propagation in viscoelastic cylindrical layered media

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Abstract

In this study, propagation of two-dimensional transient shear waves in viscoelastic cylindrical layered media is investigated. The multilayered medium consists of N different isotropic, homogeneous and linearly viscoelastic layers with two discrete relaxation times. A numerical technique which combines the Fourier transform with the method of characteristics is employed to obtain the solutions.

The numerical results are displayed in curves denoting the variations of the stress components with time at different locations. These curves reveal clearly the scattering effects caused by the reflections and refractions of waves at the boundaries and at the interfaces. The curves also display the effects of viscous damping in the wave profiles. The curves further show that the numerical technique applied is capable of predicting the sharp variations in the field variables in the neighbourhood of the wave fronts. By suitably adjusting the material constants, curves for the cases of elastic layers and viscoelastic layers with one relaxation time (standard linear solid) are also obtained. Solutions for some one-dimensional elastic and viscoelastic transient wave propagation problems are obtained as special cases. These solutions are compared with the existing ones in the literature and very good agreement is found.

1 Introduction

In this study, propagation of two-dimensional transient shear waves in cylindrical layered media consisting of N different layers is investigated. The layers of the multi-layered medium are isotropic, homogeneous and linearly
viscoelastic with discrete relaxation spectra involving two time constants. The case of elastic cylindrical layered media is treated as a special case as well. The cylindrical layered composite has a finite thickness in the radial direction; whereas, it extends to infinity in the axial direction. The inner surface of the cylindrical layered medium is subjected to axial shear tractions, while the outer surface is free of surface tractions or fixed. The layers of the composite medium are assumed to be perfectly bonded to each other. Furthermore, the multilayered medium is assumed to be initially at rest.

Method of characteristics has been employed effectively in investigating one-dimensional transient wave propagation problems in layered media. In multidimensional wave propagation problems, however, the construction of the solution by the method of characteristics becomes difficult and impractical. Hence, in this study, a numerical technique, which combines the Fourier transform with the one-dimensional method of characteristics is employed. Such a numerical technique was first introduced by Mengi and Tanrikulu [1] and later applied by Abu-Alshaikh, Turhan and Mengi [2] to investigate transient wave propagation in viscoelastic layered media with plane layers. Because of the inclusion of the method of characteristics in the analysis, the employed numerical technique is capable of describing the sharp variations of the disturbances in the neighbourhood of the wave fronts and thus it can be used conveniently for multidimensional transient wave propagation analyses.

2 Formulation of the problem

The cylindrical layered composite considered in this study has a finite thickness $H$ in the radial direction and consists of $N$ different layers. It is referred to a cylindrical coordinate system, $r, \theta, \text{ and } z$. Waves are generated in the system by applying an axial shear stress $s(\theta,t)$ at the inner surface of the layered medium. It is assumed in the formulation that the time and $\theta$ variations of $s$ are arbitrary; but it is uniform and its extent is infinite in the $z$ -direction. Thus, the problem is a two-dimensional anti-plane shear deformation problem; hence, the displacement components $u_\theta$ and $u_r$ vanish identically and the only nonvanishing displacement component $u_z$ is a function of $\theta, r \text{ and } t$, i.e., $u_z = u_z(\theta,r,t)$. Thus, the stress equation of motion for a typical layer can be written as

$$\frac{1}{r} \frac{\partial \tau_{\theta r}}{\partial \theta} + \frac{\partial \tau_{rz}}{\partial r} + \frac{1}{r} \tau_{rz} = \rho \frac{\partial v_z}{\partial t},$$

(1)

where $\tau_{\theta r}$ and $\tau_{rz}$ are the shear stress components, $v_z = \partial u_z / \partial t$ is the particle velocity component in the $z$ -direction, and $\rho$ is the mass density of the typical layer considered.
The layers of the multilayered medium are assumed to be isotropic and linearly viscoelastic. The constitutive equations for the shear stress components relevant in our study can be expressed as [3]

\[ P(D) \tau_{\theta \xi} = Q(D) \varepsilon_{\theta \xi}, \quad P(D) \tau_{\rho z} = Q(D) \varepsilon_{\rho z}, \]  

(2)

where \( \varepsilon_{\theta \xi} \) and \( \varepsilon_{\rho z} \) are the shear strain components and

\[ P(D) = \sum_{m=0}^{n} a_m D^m, \quad P(D) = \sum_{m=0}^{n} b_m D^m, \]  

(3)

in which \( a_m \) and \( b_m \) are specified constants for the typical layer considered and \( D^m = \partial^m \tau / \partial t^m \). If the initial values of \( \tau_{\theta \xi} \), \( \tau_{\rho z} \), \( \varepsilon_{\theta \xi} \), \( \varepsilon_{\rho z} \) satisfy certain conditions [3], the constitutive equations, eqns (2), can be written in terms of integral equations as

\[ \tau_{\theta \xi}(x,t) = G(t)\varepsilon_{\theta \xi}(x,0) + \int_{0}^{t} G(t-\tau) \frac{\partial \varepsilon_{\theta \xi}(x,\tau)}{\partial \tau} d\tau, \]  

\[ \tau_{\rho z}(x,t) = G(t)\varepsilon_{\rho z}(x,0) + \int_{0}^{t} G(t-\tau) \frac{\partial \varepsilon_{\rho z}(x,\tau)}{\partial \tau} d\tau, \]  

(4)

where \( G(t) \) is the shear relaxation function and \( x \) is the position vector of the particle considered. The relaxation function can be expressed in the form

\[ G(t) = 2\mu (\alpha_0 + \sum_{m=1}^{n} e^{-t/\tau_m}), \]  

(5)

where \( \mu = G(0)/2 \) is the shear impact modulus, \( \tau_m \) are positive time constants and \( \alpha_m \) are positive non-dimensional constants such that their sum is equal to one.

In this study, the viscoelastic solid is modelled with \( n = 2 \) in eqns (3) and (5). Constitutive equations of the same form were used in various one-dimensional transient wave propagation problems in viscoelastic media [4-5]. Equations (3) and (5) with \( n = 1 \) represent the constitutive equations for a
standard linear solid. The constants in eqns (3) and (5) with \( n = 2 \) are related according to [4]

\[
a_0 = (\tau_1 \tau_2)^{-1}, \quad a_1 = (\tau_1 + \tau_2)/\tau_1 \tau_2, \quad a_2 = 1, \quad b_0 = (2\mu \alpha_0 / \tau_1 \tau_2),
\]

\[
b_1 = 2\mu [\tau_1 (\alpha_0 + \alpha_1) + \tau_2 (\alpha_0 + \alpha_2)] / \tau_1 \tau_2, \quad b_2 = 2\mu.
\]

The shear strain components are related to \( u_z \) through

\[
\varepsilon_{\theta z} = \frac{1}{2r} \frac{\partial u_z}{\partial \theta}, \quad \varepsilon_{rz} = \frac{1}{2} \frac{\partial u_z}{\partial r}.
\]

The formulation of the problem is completed by stating the boundary, initial and interface conditions. The boundary condition at the inner surface \( r = r_i \) of the multilayered medium is

\[
\tau_{rz}(\theta, r_i, t) = s(\theta, t),
\]

where \( s(\theta, t) \) is a prescribed function of \( \theta \) and \( t \). The outer surface \( r = r_o \) is either free of surface tractions or fixed. Hence, the boundary condition can be written as

\[
\tau_{rz}(\theta, r_o, t) = 0 \quad \text{or} \quad u_z(\theta, r_o, t) = 0.
\]

The multilayered medium is assumed to be initially at rest; hence, all the field variables are zero at \( t = 0 \). The layers of the multilayered medium are assumed to be perfectly bonded to each other. Hence, the interface conditions imply that \( \tau_{rz} \) and \( u_z \) are continuous across the interfaces of the layers.

The formulation of the problem is thus complete. The governing field equations, eqns (1-2) and (7), will now be applied to each layer and the solutions will be required to satisfy the continuity conditions at the interfaces, the boundary conditions at the inner and outer surfaces, eqns (8-9), and zero initial conditions.

### 3 Solution of the problem

The solution is obtained by employing a numerical technique that combines the Fourier transform with the method of characteristics. The technique involves first the application of the Fourier transform to the governing equations over the space variable \( \theta \), then the integration of the resulting one-dimensional hyperbolic equations by the method of characteristics, and finally inverting the
solution back into real space. For this purpose, we first write the constitutive equations for the typical layer considered, eqns (2-3) with $n = 2$ and $a_2 = 1$, in view of eqns (7), as

$$
a_0 \tau_{\theta \theta} + a_1 \tau_{r \theta} + \frac{\partial T_{\theta \theta}}{\partial t} = b_0 \varepsilon_{\theta \theta} + b_1 E_{\theta \theta} + \frac{b_2}{2r} \frac{\partial A_z}{\partial \theta},
$$

$$
a_0 \tau_{r r} + a_1 \tau_{r z} + \frac{\partial T_{r r}}{\partial t} = b_0 \varepsilon_{r r} + b_1 E_{r r} + \frac{b_2}{2} \frac{\partial A_z}{\partial r},
$$

where

$$
T_{\theta \theta} = \frac{\partial \tau_{\theta \theta}}{\partial t}, T_{r r} = \frac{\partial \tau_{r r}}{\partial t}, E_{\theta \theta} = \frac{\partial \varepsilon_{\theta \theta}}{\partial t}, E_{r r} = \frac{\partial \varepsilon_{r r}}{\partial t}, A_z = \frac{\partial v_z}{\partial t}. \tag{11}
$$

Note that $A_z$, as defined in eqn (11), is the particle acceleration in the $z$-direction. Furthermore, we have the compatibility equations

$$
\frac{\partial E_{\theta \theta}}{\partial t} - \frac{1}{2r} \frac{\partial A_z}{\partial \theta} = 0, \quad \frac{\partial E_{r r}}{\partial t} - \frac{1}{2} \frac{\partial A_z}{\partial r} = 0. \tag{12}
$$

Equations (1) and (10-12) constitute a system of first-order governing partial differential equations, which can be written in matrix form as

$$
A \mathbf{U}_t + B \mathbf{U}_r + D \mathbf{U}_\theta + \mathbf{F} = 0, \tag{13}
$$

where $A = I$, an $(11 \times 11)$ identity matrix, and $B$ and $D$ are $(11 \times 11)$ square matrices, $F$ is an eleven-dimensional column vector and $U$ is an eleven-dimensional column vector defined by

$$
\mathbf{U} = (T_{\theta \theta}, T_{r r}, \tau_{\theta \theta}, \tau_{r r}, E_{\theta \theta}, E_{r r}, \varepsilon_{\theta \theta}, \varepsilon_{r r}, u_z, v_z, A_z)^T, \tag{14}
$$

where $T$ designates the transpose. In eqn (13), comma denotes partial differentiation, i.e., $\mathbf{U}_t = (\partial \mathbf{U} / \partial t)$, $\mathbf{U}_\theta = (\partial \mathbf{U} / \partial \theta)$, $\mathbf{U}_r = (\partial \mathbf{U} / \partial r)$.

To apply the numerical technique stated above, we first take the Fourier transform of the system of governing equations, eqn (13), with respect to $\theta$. This eliminates the dependence of the field variables on $\theta$, and the resulting transformed equations can be written in the form of a system of first-order partial differential equations as

$$
A \mathbf{U}_t^F + B \mathbf{U}_r^F + C = 0, \tag{15}
$$
where the superscript $F$ stands for the Fourier transform. The transform equations, eqns (15), involve the Fourier transform parameter $k$ and $U^F$ is defined as in eqn (14) with the field variables replaced by their Fourier transforms.

The Fourier transform of the boundary conditions, eqns (8-9), with respect to $\theta$ gives

$$\tau_{rz}^F(k, r_i, t) = s^F(k, t) \quad \text{and} \quad \tau_{rz}^F(k, r_0, t) = 0 \quad \text{or} \quad U_z^F(k, r_0, t) = 0. \quad (16)$$

The Fourier transforms of the interface conditions require that the transformed shear stress $\tau_{rz}^F$ and displacement $U_z^F$ be continuous across the interfaces of the layers. The formulation of the problem in the Fourier transform space is thus complete.

The second step of the procedure involves the solution of the problem in the Fourier transform space, which requires the solution of eqns (15) for each layer satisfying the boundary conditions, eqns (16), at the boundaries, the interface conditions at the interfaces and the zero initial conditions. The system of governing equations, eqns (15), is hyperbolic, and the solution for a given value of the wave number (transform parameter) $k$ is constructed by employing the method of characteristics.

Finally, we invert numerically the solution obtained in the Fourier space back into the real space by employing the inverse Fourier transform relation. The numerical inversion requires the construction of the solution in the characteristic plane at the discrete wave number points $(k_0, k_1, k_2, \ldots)$ with an increment $\Delta k$. The number of wave number points considered and the cut-off value of $k$ to be considered in the analysis should be chosen properly, since they play an important role in achieving a desired accuracy in the solution. The inversion is performed conveniently by using the fast Fourier transform (FFT) algorithm [6]. It may be noted that the same algorithm can also be used for computing the Fourier transform $s^F(k, t)$ of the applied shear stresses.

4 Numerical results and discussions

Throughout this study, the numerical computations and the results are displayed in terms of non-dimensional quantities. The non-dimensional quantities are denoted by putting bars over them. The characteristic quantities used in non-dimensionalization are: $\rho_1$, the mass density of the innermost layer, $H$ the thickness of the layered medium, $c_1 (= \sqrt{\mu_1 / \rho_1})$ and $\mu_1 (= G_1 (0)/2)$ the shear wave velocity and the shear impact modulus of the innermost layer,
respectively. Moreover, the results are obtained when the time and $\theta$ variations of $s$ applied at the inner surface $r = r_i$ are in the form

$$s(\theta, t) = s_0 f(\theta) g(t),$$

where $s_0$ is the intensity of the applied load, which is chosen in the computations as $s_0 = (s_0 / \mu) = 1$. In eqn (17), $g(t)$ is taken as a unit step function with an initial ramp.

The computer program has been written for viscoelastic cylindrical composites consisting of $N$ different layers with two discrete relaxation times ($n=2$), whereas, the results for the elastic and viscoelastic standard linear solid are found as special cases as well. The numerical results are obtained for composites consisting of $3$-pairs of alternating layers, layer 1 and layer 2, starting from the innermost layer in a sequence $1/2/1/2/1/2$. In the numerical examples two types of composites are considered, and four different problems are solved all consisting of $3$-pairs of alternating layers. The common material and geometric properties of the two composites are given in Table 1, where $\bar{h}$ denotes the non-dimensional thickness of the layer in the corresponding composite material.

<table>
<thead>
<tr>
<th>Table 1. The common material and geometric properties of the two composites.</th>
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<tbody>
<tr>
<td>Viscoelastic Composite 1</td>
</tr>
<tr>
<td>Layer 1</td>
</tr>
<tr>
<td>$\bar{h}$</td>
</tr>
<tr>
<td>1/9</td>
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<tr>
<td>Layer 2</td>
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<tr>
<td>2/9</td>
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<tr>
<td>Viscoelastic Composite 2</td>
</tr>
<tr>
<td>Layer 1</td>
</tr>
<tr>
<td>1/9</td>
</tr>
<tr>
<td>Layer 2</td>
</tr>
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<td>2/9</td>
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</tbody>
</table>

In the first two problems, the inner surface of the composite bodies ($r_i = 1$) is assumed to be subjected to uniform shear tractions in $\theta$-direction and the outer surface ($r_o = 2$) in these problems as well as in the other two problems is assumed to be free of tractions. The problem is treated as a two-dimensional multilayered viscoelastic problem. In Figures 1 and 2, the variations of $\tau_{\tau}/s_0$ with non-dimensional time at the location $r = 1.21$ are displayed for the elastic case and two viscoelastic composites with one relaxation time (standard linear solid) obtained as special cases. In Figure 1, for the two viscoelastic composites, the properties of the layer 2 are taken the same as those of layer 1. Hence, the curves of Figure 1 represent solutions for a single elastic or viscoelastic layer. Since the surface tractions applied at the inner surface are assumed to be uniform, the curves obtained from the two-dimensional solution basically
represent one-dimensional solutions. The curves of Figure 1 clearly reveal the effects of reflections from the inner and outer boundaries and the effects of geometric dispersion. The curves representing the viscoelastic solutions in Figure 1 clearly display the effects of viscous damping, more pronounced for smaller value of $\alpha_0$, in the wave profiles. The curves denote that the attenuation in the wave profiles gets larger as time increases. The curves of Figure 2 pertain to composites consisting of three pairs of alternating layers with properties given in Table 1. The layers of the viscoelastic composites are modelled as standard linear solid (one relaxation time). These curves exhibit similar trends; the reflections and refractions from the interfaces are pronounced clearly. The curves of Figure 2 obtained from a two-dimensional are compared with those obtained from a one-dimensional formulation by Turhan & Abu-Alshaikh [7]; perfect agreement is found. The forms of the curves in Figures 1 and 2 further show that the numerical method employed in this study is capable of predicting the sharp variations at the wave fronts.

In Figure 3, the variation of $\tau_{rz}/s_0$ with time $\tilde{t}$ at the location $\theta = 0^\circ$, $\tilde{r} = 1.21$ of a single layer is displayed. The material properties are taken as given in Table 1 with $\bar{r}_2 = (\bar{r}_1/2)$, $\alpha_1 = \alpha_2$ such that $\alpha_0 + \alpha_1 + \alpha_2 = 1$ and the properties of layer 2 being the same as those of layer 1. Note that these properties correspond to viscoelastic layers with two discrete relaxation times ($n=2$). The solution for the elastic case is obtained and displayed in the figure as well. Furthermore, the inner surface is partially subjected to trapezoidal distribution of axial shear tractions in the circumferential direction. This distribution is equal unity in the intervals $|\theta| \leq (3\pi/16)$ and $(13\pi/16) \leq |\theta|$ and falling linearly to zero in the range of interval $(\pi/4) \leq |\theta| \leq (3\pi/4)$. The curves
of Figure 3 clearly show the effects of reflections at the inner and outer surfaces, geometric dispersion and attenuation in the wave profiles due to viscous damping, more pronounced for smaller values of $\alpha_0$. Furthermore, in the curves of Figure 3, the scattering effects are seen which are not apparent in the curves of Figure 1, which basically represent a one-dimensional wave propagation problem.

![Figure 2: Time variation of $\tau_{rz}/s_0$ at $\bar{r}=1.21$ for multilayered media.](image)

![Figure 3: Time variation of $\tau_{rz}/s_0$ at $\theta=0^\circ$ and $\bar{r}=1.21$ for a single layer, where the inner surface is partially loaded.](image)

Finally, in Figure 4, the time variation of $\tau_{rz}/s_0$ at the location $\theta=0^\circ$, $\bar{r}=1.21$ is displayed for multilayered viscoelastic composites 1 and 2 consisting of three pairs of alternating layers. The material properties are as given in
Table 1 with $\bar{\tau}_2 = (\bar{\tau}_1 / 2)$ and $\alpha_1 = \alpha_2$ such that $\alpha_0 + \alpha_1 + \alpha_2 = 1$. The elastic solution obtained as a special case is also given in the figure. The curves of Figure 4 display similar trends. In addition, these curves reveal the effects of reflections and refractions at the interfaces of the layers. We further observe that the viscous effects smooth out the response and become more pronounced as time increases.

![Graph](image)

Figure 4: Time variation of $\tau_{zz}/s_0$ at $\theta = 0^\circ$ and $r = 1.21$ for multilayered media, where the inner surface is partially loaded.

References


