Stochastic modelling in aeroelasticity

J. Náprstek
Institute of Theoretical and Applied Mechanics,
Academy of Sciences of the Czech Republic

Abstract

The paper investigates the stochastic stability of a cross-wind movement influenced by parametric noises generated by the interaction of a moving bluff body and an air flow. Conditions of the loss or resumption of movement stability by means of the stochastic version of Lyapunov function are derived. Subcritical and various types of the supercritical modes of system response combining the deterministic and the stochastic response components are investigated. These effects are verified by several mathematical models of non-linear damping. There has been shown a different response sensitivity to the mathematical model in the individual domains of stochastic instability which makes possible to explain some paradoxes known from wind channel tests.

1 Introduction

Slender civil engineering structures are subjected to dynamic excitation due to air flow. As regards cross-wind effects it seems that galloping and flutter are the most dangerous. Both these effects are considered as a certain type of a stability loss of the structure response. This paper is aimed at the galloping. It predominantly originates in the effects of self-excitation while the later one is rather corresponding with an influence of non-conservative aeroelastic forces. When only the approximative position of bifurcation points is to be determined, the linear formulation of the problem is satisfactory. Despite of that the problem should be formulated in a nonlinear state as apart from the determination of a critical air flow velocities it is necessary to examine also the response characteristics in the post-critical state.

The cross-wind movement of a body is influenced significantly by non-linear terms and by fluctuations of the ambient medium. These fluctuations are mostly of random character. The principal task is to describe the individual response types and to determine conditions of their stochastic
stability.

The problem of the transverse movement of a body in an air flow, modelled by a system with one degree of freedom, was dealt with by a number of authors in the past. The first papers were concerned with partial problems using linear deterministic models, e.g. [1], [2]. Other articles noted the stochastic character of some phenomena, e.g. [3], [4], [5]. Probably the first attempt at a systematic description of these phenomena by methods of stochastic mechanics was the paper [6] respecting the non-linear character of damping influenced by random noises.

2 Model of the van der Pol type

Dealing with cross wind vibration of the galloping type, e.g. [3], the motion can be modelled by an SDOF system using a nonlinear differential equation:

$$M \cdot \ddot{u}(t) + F_{\text{dam}}(u, \dot{u}) + C \cdot u(t) = \varphi_M(t)$$

(1)

In spite of a great number of studies concerned with the problem, an opinion regarding the structure of the damping force is by far not fixed. The same sources present the damping force in different forms and compare the effects of the variability of the mathematical model on the properties of the system.

Let us assess two typically applied models. The first one will be assessed in this chapter. The force $F_{\text{dam}}(u, \dot{u})$ is introduced being dependent on $u^2(t)$ according to [7]:

$$F_{\text{1 dam}}(u, \dot{u}) = \left[ 2M \cdot (\omega_{bc} + \frac{1}{2} \gamma^2 \cdot u^2(t)) - 2M \omega_{ba} \right] \cdot \dot{u}(t)$$

(2)

$2M \cdot (\omega_{bc} + \frac{1}{2} \gamma^2 \cdot u^2(t))$ - amplitude-dependent damping "parameter"; for small amplitudes represents classical linear constant viscous damping $2M \omega_{bc}$, as the quadratic term is negligible; this damping "parameter" increases for larger amplitudes and can act as a stabilising factor;

$2M \omega_{ba} \cdot \dot{u}(t)$ - the effect of a change of lift force due to the variation of the angle of attack, which has been caused by the vibration of the body perpendicularly to the flow. With aerodynamically unstable body cross section this term is negative, i.e. this component of damping represents the self-excitation.

Eq.(1) can be understood either as a really one dimensional idealisation of the problem or as a description of the movement of a bar fixed on both ends in a frequency range corresponding with the first natural mode.

In this way Eq.(1) acquires van der Pol form and corresponds with the equation given in [8]. The parameter $\omega_{ba}$ can approximate as a linear increasing function of nominal air flow velocity. That means there is a certain critical mean air flow velocity when the parameter $\omega_{ba}$ attains the value of $\omega_{bc}$ and the effective damping value will equal zero. Under these circumstances the system in linear state would lose stability in this moment. When the response amplitudes begin markedly rising, the nonlinear damping component will come to the force the influence of which will rise progressively with increasing displacements.
The parameter of the linear damping component $\omega_b = \omega_{ba} - \omega_{bc}$ and the natural frequency of the linearised system $\omega_0^2 = C/M$ will be influenced considerably in the air flow environment with heavy time-dependent noises. That means that, respecting (2), Eq. (1) will acquire the following initial form, see [6]:

$$\ddot{u}(t) - (2\omega_b - \gamma^2 \cdot u^2(t) + w_2(t)) \cdot \dot{u}(t) + (\omega_0^2 + w_1(t)) \cdot u(t) = 0$$

$$\omega_0^2 = C/M; \quad \omega_b = \omega_{ba} - \omega_{bc}.$$  

$w_1(t), w_2(t)$ - Gaussian white noises of intensities $s_{11}, s_{22}$ and cross intensity $s_{12}$.

Introducing the phase space $\mathbf{u} = \{u_1 = u, u_2 = \dot{u}\}$, Eq. (3) can be written in Ito's form:

$$\dot{u}_1 = u_2; \quad \dot{u}_2 = (2\omega_b - \gamma^2 \cdot u_1^2 + u_2) \cdot u_2 - (\omega_0^2 + w_1) \cdot u_1$$

### 3 Model of the Rayleigh type

The second most frequently applied damping force model is based on the nonlinear component dependent on the square of displacement velocity $\dot{u}^2(t)$, see e.g. [7], [8]:

$$F_{2,\text{dam}}(u, \dot{u}) = \left[ 2M \cdot (\omega_{bc} + \frac{1}{2} \delta^2 \cdot \dot{u}^2(t)) + 2M \omega_{ba} \right] \cdot \dot{u}(t)$$

This damping force model results in an equation of motion of the Rayleigh type. However, even the wind tunnel tests did not succeed in giving an explicit answer which version should be preferred.

Eq. (5) corresponds probably best with the theoretical considerations of the aerodynamic characteristics of the air flow and a transversally moving body. This applies primarily to the interaction of the frontal pressure (generating also the cross component) and the uplift coefficient. Both depend on apparent cross section rotation and can be described by Theodorsen formulæ [8]. Taking into account only the linear and the cubic terms of its Taylor series, Eq. (5) can be obtained. Like in Eq. (2) also in Eq. (5) the parameters $\omega_{ba}, \gamma, \delta$ contain such factors as mean flow velocity, density of air, aerodynamic admittance, characteristic cross section dimension, etc.

Introducing parametric noises into the damping force and the effective stiffness in the same way as in the first case, the Ito's system acquires a somewhat different form:

$$\dot{u}_1 = u_2; \quad \dot{u}_2 = (2\omega_b - \delta^2 \cdot u_1^2 + u_2) \cdot u_2 - (\omega_0^2 + w_1) \cdot u_1$$

The linear part of the damping is the same as in Eq. (8), the nonlinear stabilising factor is realised through the square of velocity instead of the square of displacement. It can be expected that these two models lead to
qualitatively different results in non-stationary regimes while the differences in stationary states, linear as well as nonlinear, are rather quantitative.

4 Stability domains

Let us try to estimate the stability domains of both models described by systems (4) and (6) using the second Lyapunov method, e.g. the Lyapunov function, in stochastic approach.

It can be shown that the significance of time derivative of a positive definite function (which should be Lyapunov function) in the deterministic domain, see e.g. [9], is taken over by adjoined FPK operator \( L\{\cdot\} \) in stochastic domain, see e.g. [10], [11]:

\[
L\{\lambda(t, u)\} = \frac{\partial \lambda(t, u)}{\partial t} + \sum_{i=1}^{n} \frac{\partial \lambda(t, u)}{\partial u_i} \kappa_i + \frac{1}{2} \sum_{i,j=1}^{n} \frac{\partial^2 \lambda(t, u)}{\partial u_i \partial u_j} \kappa_{ij}
\]

(7)

If \( \psi = L\{\lambda(t, u)\} \) is negative, the system is stable in probability, see e.g. [10], [12]. If the stochastic stability of the system:

\[
\dot{u}_i = f_i(u) + \sum_{k=1}^{m} h_{ik}(u) \cdot w_k(t) ; \quad u(t_o) = u_o
\]

(8)

should be investigated, the diffusion coefficients \( \kappa_i, \kappa_{ij} \) have the form:

\[
\kappa_i = f_i(u) ; \quad \kappa_{ij} = \sum_{k=1}^{m} h_{ik}(u) h_{ji}(u) \cdot s_{kl}
\]

(9)

The symbols used in Eqs (7), (8), (9) have the following meaning:

- \( u \) - vector of phase variables (\( n \) elements);
- \( s_{kl} \) - intensities of white noises \( w_k(t)(k, l = 1, \ldots, m) \);
- \( h_{ik}(u) \) - continuous differentiable functions.

The system (4) has the first integral of total energy type. The principal form of Lyapunov function can be selected as follows:

\[
\lambda(t, u) = \frac{1}{2}(u_2 + G_1(u_1))^2 + G_2(u_1)
\]

(10)

where we have denoted:

\[
G_1(u_1) = \int_{0}^{u_1} (2\omega_b - \gamma^2 \cdot \xi^2) \cdot d\xi = 2\omega_b u_1 - \frac{1}{3} \gamma^2 \cdot u_1^3;
\]

(11)

\[
G_2(u_1) = \int_{0}^{u_1} \omega_0^2 \zeta \cdot d\xi = \frac{1}{2} \omega_0^2 u_1^2
\]

In the function \( \lambda(t, u) \) there are terms representing the influence of kinetic energy \( u_2^2 \) and that of potential energy \( G_2(u_1) \) and a certain function \( G_2(u_1) \) supplementing the influence of potential energy.
With reference to Eqs. (8), (9) in the previous paragraph and with regard to Eqs (4) it holds that $n = m = 2$ and:

$$ f_1(u) = u_2 : \quad f_2(u) = (2\omega_b - \gamma^2 \cdot u_1^2) \cdot u_2 - \omega_0^2 \cdot u_1 $$

$$ h_{11}(u) = h_{12}(u) = 0 ; \quad h_{21}(u) = -u_1 : \quad h_{22}(u) = -u_2 $$

Hence it follows that:

$$ \kappa_1 = u_2 : \quad \kappa_2 = (2\omega_b - \gamma^2 \cdot u_1^2) \cdot u_2 - \omega_0^2 \cdot u_1 $$

$$ \kappa_{11} = \kappa_{12} = \kappa_{21} = 0 ; \quad \kappa_{22} = u_1^2 s_{11} + u_1 u_2 (s_{12} + s_{21}) + u_2^2 s_{22} $$

Lyapunov function (10) is obviously positive definite, as soon as the condition

$$ G_2(u_1) = \frac{1}{2} \omega_0^2 u_1^2 > 0 : \quad u_1 \neq 0 $$

has been complied with.

The condition (12) is complied with reference to (11). Consequently, function (10) is positive, if at least one of the phase coordinates $u_1, u_2$ is different from zero. As in the same time it equals zero only for $u_1, u_2 = 0$, it is positive definite within the whole phase space. Moreover, function (10) has an infinitely large lower limit, as

$$ \lim_{u_1 \to \infty} \lambda(t, u) = \infty $$

The derivative of the function $\lambda(t, u)$ with respect to phase variables:

$$ \frac{\partial \lambda(t, u)}{\partial u_1} = (u_2 - 2\omega_b u_1 + \frac{1}{3} \gamma^2 u_1^3) (-2\omega_b + \gamma^2 u_1^2) + \omega_0^2 u_1 : $$

$$ \frac{\partial \lambda(t, u)}{\partial u_2} = u_2 + (-2\omega_b u_1 + \frac{1}{3} \gamma^2 u_1^3) ; \quad \frac{\partial^2 \lambda(t, u)}{\partial u_2^2} = 1 . $$

Using these partial expressions and Eq. (7), the $L\{\lambda(t, u)\}$ can be determined:

$$ \psi(t, u) = L\{\lambda(t, u)\} = $$

$$ -\omega_0^2 u_1^2 (-2\omega_b + \frac{1}{3} \gamma^2 u_1^2) + (u_1^2 s_{11} + u_1 u_2 (s_{12} + s_{21}) + u_2^2 s_{22}) $$

Let us examine now function (13). There are domains in the phase space $u_1, u_2$ where it is negative and others where it is positive. This gives rise to the conditions under which the system is stable.

Let us deal with a stochastic case when the noises $w_1, w_2$ are independent, i.e. when $s_{12} = s_{21} = 0$. It follows from the basic structure of Eq. (13) that in contrast to the deterministic case both noises exercise a destabilising effect. Even in the case of subcritical air flow velocity (in the deterministic meaning), the system may lose exponential stability in the stochastic meaning, if it does not hold that:

$$ \psi(t, u) = -\omega_0^2 u_1^2 (-2\omega_b + \frac{1}{3} \gamma^2 u_1^2) + (u_1^2 s_{11} + u_2^2 s_{22}) < 0 $$

(14)
The equation \( \psi(t, u) = 0 \) can be used for the determination of an estimate of the boundaries of the exponential stability of the initial system (1). As the system contains only symmetrical parametric noises and no other excitation sources, its response will be symmetrical, too. The processes \( u_1, u_2 \) can be considered centered. If we apply the mathematical mean value operator to this equation we obtain, after modification:

\[
M_\psi = -\frac{1}{3} \omega_0^2 \gamma^2 \cdot D_{11}^4 + (2\omega_0^2 \omega_b + s_1)D_{11}^2 + s_{22} D_{22}^2 = 0
\]  

(15)

where \( D_{11}^2, D_{11}^4 \) are the second and fourth respectively central moments of the process \( u_1 \) and \( D_{22}^2 \) is the second central moment of the process \( u_2 \). The symbol \( M_\psi \) denotes the mathematical mean value of the stochastic function \( \psi(t, u) \). Eq. (15) describes the stability limit. If we are able, with regard to the type of probability density of the process \( u_1 \), to define the relation between \( D_{11}^4 \) a \( D_{11}^2 \), we can consider (15) to be a figure in the plane \( D_{11}^2, D_{22}^2 \), where only the first quadrant, i.e. \( D_{11}^2 > 0, D_{22}^2 > 0 \) is meaningful.

If the process \( u_1 \) can be considered at least approximately Gaussian, then it holds that \( D_{11}^4 = 3(D_{11}^2)^2 \). This approximation is probably possible with regard to the fact that the stiffness characteristic of the system (3) is linear. In such a case the estimated stability boundary will have the form of

\[
-\frac{\omega_0^2 \gamma^2}{s_{22}} \cdot (D_{11}^2)^2 + \frac{(2\omega_0^2 \omega_b + s_1)}{s_{22}} D_{11}^2 + D_{22}^2 = 0
\]  

(16)

The expression (16) represents a parabola with the vertical axis and passing through the origin, see Fig. 1. The stable state is due to the negative value of the function \( \psi(t, u) \), i.e. between the parabola and the positive horizontal axis when \( D_{11}^2 > 0 \). Consequently, three different cases can occur. If the vertex is situated left of the axis of \( D_{22}^2 \), the stability domain

![Figure 1: Stability domains of a system with displacement influenced by non-linear damping; 1 - sub-critical, 2 - critical, 3 - super-critical flow speed](image-url)
reaches the origin, so that stability of the system (3) can be expected. This applies, when:

\[ 2\omega_0^2\omega_b + s_{11} < 0 \]  

(17)

The condition (17) imposes the requirement of the negative value of \( \omega_b \) reduced by the effect of \( s_{11} \) in contrast to the deterministic case; consequently, the condition is more strict in the stochastic domain. We can say that the condition (17) defines the generalised criterion for critical air flow velocity with reference to a certain parametric noise level.

If the vertex of the parabola coincides with the origin, i.e. if the inequality (17) becomes an equation, the system is still stable in the environs of the origin. Negative or at least zero values of \( v(t, u) \) can be obtained without any of the quantities of \( D_{11}, D_{22}^2 \) being higher than zero. In that case, however, the stability is merely in probability.

If the vertex of the parabola is situated right of the axis \( D_{22}^2 \), stable state can take place only if \( D_{11}^2 > 0 \), i.e. if it holds at least that

\[ D_{11}^2 > \frac{2\omega_0^2\omega_b + s_{11}}{\omega_0^2\gamma^2} ; \text{ point } D_{1k} \]  

(18)

This state can taken place also with a negative \( \omega_b \), if \( s_{11} \) is of sufficient magnitude. Consequently, the system is always capable of attaining secondary stability, but at the cost of amplitudes different from zero, even if finite and determinable, described e.g. by stochastic moments. However, the stability corresponds only with stability in probability and, therefore, it is the lowest of all above mentioned stability types.

Let us return to Eq. (6) modelling nonlinearity in damping as a function of velocity. The basic form of Lyapunov function represents once again the first integral of the system total energy

\[ \lambda(t, u) = \frac{1}{2}u_2^2 + G_2(u_1) \]  

(19)

where it has been denoted:

\[ G_2(u_1) = \int_0^{u_1} \omega_0^2\zeta \cdot d\zeta = \frac{1}{2}\omega_0^2u_1^2 \]  

(20)

Like in the preceding case the \( L\{\lambda(t, u)\} \) ascertaining a decrease or increase of the system (6) energy along its trajectory will be determined.

Let us proceed according to Eqs. (8), (9) \( (n = m = 2) \):

\[ f_1(u) = u_2 ; \quad f_2(u) = (2\omega_b - \delta^2 \cdot u_2^2) \cdot u_2 - \omega_0^2 \cdot u_1 \]

The functions \( h_{ij}(u) \) are the same as previously:

\[ h_{11}(u) = h_{12}(u) = 0 ; \quad h_{21}(u) = -u_1 ; \quad h_{22}(u) = -u_2 \]

Hence we obtain the diffusion coefficients:

\[ \kappa_1 = u_2 ; \quad \kappa_2 = (2\omega_b - \delta^2 \cdot u_2^2) \cdot u_2 - \omega_0^2 \cdot u_1 \]
Figure 2: Comparison of stability domains of two systems with different model of non-linear damping; (a) significantly sub-critical flow speed; (b) super-critical flow speed

\[ \kappa_{11} = \kappa_{12} = \kappa_{21} = 0; \quad \kappa_{22} = u_1^2 s_{11} + u_1 u_2 (s_{12} + s_{21}) + u_2^2 s_{22} \]

With reference to (7) we shall obtain:

\[ \frac{\partial \lambda(t, u)}{\partial u_1} = \omega_0 u_1; \quad \frac{\partial \lambda(t, u)}{\partial u_2} = u_2; \quad \frac{\partial^2 \lambda(t, u)}{\partial u_2^2} = 1 \]

By summation of preceding expressions we obtain, after modification:

\[ \psi(t, u) = L\{\lambda(t, u)\} = u_2^2 (2 \omega_b - \delta^2 u_2^2) + u_1^2 s_{11} + u_1 u_2 (s_{12} + s_{21}) + u_2^2 s_{22} \]

The condition of positive definiteness of Lyapunov function (3) is the same as in the preceding case:

\[ G_2(u_1) > 0; \quad u_1 \neq 0 \quad (22) \]

and, consequently, is complied with. In both cases it imposes certain requirements on the characteristic of system stiffness. Although it has been introduced as linear, it is obvious that the condition (12) or (22) would be complied with not only by any stiffening characteristic, but also any softening characteristic, if described by an increasing function within the whole interval \( u_1 \epsilon(-\infty, \infty) \). That means that the system must not permit any phenomena of "snap through" type.

We shall use the equation \( \psi(t, u) = 0 \) also this time for the determination of the boundaries of exponential stability. If we limit our considerations, once again, to independent noises \( w_1, w_2 \), we obtain by the application of the mathematical mean value operator and subsequent modification:

\[ M_\psi = -\delta^2 D_{22}^4 + 2\omega_b D_{22}^2 + s_{11} D_{11}^2 + s_{22} D_{22}^2 = 0 \quad (23) \]

If we carry out the obvious commutation in Eq. (23), it will acquire the same structure as Eq. (15). By adopting the hypothesis of approximately Gaussian response, i.e. \( D_{22}^4 = 3(D_{22}^2)^2 \), we obtain the equation:

\[ -3\delta^2 (D_{22}^2)^2 + (2\omega_b + s_{22}) D_{22}^2 + D_{11}^2 = 0 \quad (24) \]
representing a parabola with the horizontal axis being opened towards positive values of $D^2_{11}$. The analysis of its characteristics is similar to the preceding case. It means, if it holds that:

$$2\omega_b + s_{22} < 0 \quad (25)$$

the vertex of the parabola is below the horizontal axis and the system is exponentially stable. The transition state characterised by asymptotic stability will occur when the vertex is situated in the origin. In such a case the inequality (25) changes into an equation. If the expression on the left-hand side of (25) is positive, the vertex of the parabola is situated above the horizontal axis. The system acquires secondary stability thanks to the nonlinear term and vibrates within a certain band of the non-zero breadth with the stability in probability. In case of supercritical air flow velocity in the meaning of the condition (25), the stability domain adjoins the vertical axis of $D^2_{22}$ beginning with the point:

$$D^2_{22} = \frac{2\omega_b + s_{22}}{3d^2}; \quad \text{point } D_{2k} \quad (26)$$

Let us compare now both cases of nonlinearity, see Fig. 2. Considering the fact that the movement of the system at sub-critical velocity is of very narrow band character and appears rather as a sinusoid with mild perturbations, see [6], we can assess $s_{11}$ as $\omega_b^2 s_{22}$. In such a case the conditions (18) and (26) appear practically the same, except for the multiplicative constant. The mechanism of the loss of stability and regained stability at super-critical velocity are in both cases principally the same.

While the air flow velocity and the value of $\omega_b$ remains so low that the intersection of stability domains extends as far as the origin, see Fig. 2(a), the system is stable in probability and perturbations of displacements due to parametric noises do not differ practically. We can say that the behaviour of the system is not visibly influenced by the selection of the damping model.

The domain of instability common to both models originates in supercritical regime. The response structure in these cases depends considerably on the damping model. However, with increasing air flow velocity we move mostly along the diagonal in the first quadrant in the Fig. 2(b) and, beginning with a certain velocity (point $D_0$), we shall reach again the domain of stability common to both models. This domain is characterised again by the dropping dependence on the type of the nonlinear damping force model.

5 Conclusion

The theoretical models of galloping influenced by parametric noises show a considerable sensitivity of a system to the initiation of self excited random vibrations with a significant deterministic component the frequency of which approaches the natural frequency of the system. The system response is substantially influenced in post critical regimes by nonlinear terms entering the damping force. The existence of decisive boundaries of various regimes and stability levels are not influenced much by the selection of the mathematical model of nonlinear damping force. The differences are rather quantitative. In principle it is possible to speak about the sub-critical regime when the
system is in the state of stochastic stability in probability of the trivial solution. In super-critical regime the system acquires secondary stability due to the influence of nonlinear terms.

A comparison with direct methods (e.g. the harmonic balance method) reveals that the application of Lyapunov function results in stronger stability conditions. That means that Lyapunov function leads to more conservative results. However, it is an usual effect dealing with nonlinear systems. An indubitable advantage of Lyapunov function, however, consists in the fact that it provides a much better general overview about the behaviour of a system both in the deterministic and in the stochastic domains.

Generally speaking it is possible to say that parametric noises introduced into the linear part of the damping forces will manifest themselves by a reduction of effective damping consisting of the basic viscous damping, aeroelastic damping and effect of parametric noises.

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References