Steady and unsteady flow solutions using velocity singularities for fixed and oscillating aerofoils and wings

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Abstract
This paper presents several methods of solutions recently developed based on velocity singularities for the analysis of steady and unsteady flows past fixed and oscillating aerofoils and wings. All these methods led to efficient theoretical solutions in closed form, which distinguishes them from the previous solutions, in comparison with which they were validated. The method for the unsteady flows past oscillating aerofoils permitted to obtain closed form solutions for both cases of rigid aerofoil oscillations and of flexural oscillations. The method for the wings of finite span permits to obtain efficient solutions for more general problems of practical interest, such as those involving spanwise variable incidence. The non-linear solutions for flows past aerofoils of symmetrical thickness were found in very good agreement with the experimental results; to the author’s best knowledge these are the first non-linear theoretical solutions obtained in closed form.

1 Introduction
The steady flow around aerofoils (or wing sections) and finite span wings has been extensively studied during the twenties century. First, theoretical solutions have been obtained by conformal transformation for specific shapes of aerofoils, such as Joukowski, Karman-Trefftz, Betz-Keune, Müller, von Mises and Carafoli aerofoils [1, 4, 20]. For more general aerofoil shapes used in the aeronautical applications, such as NACA aerorfoils, several linear theoretical solutions have been developed by various authors, such as the classical solution obtained by modified Taylor expansions for thin aerofoils (Birnbaum-Glauert [4, 20]), and the closed form solutions based on velocity singularities developed by Mateescu et al.
Other solutions involving intensive numerical calculations have been obtained using conformal mappings (such as Sells [21], Halsey [6], Ives [9], Bauer, Garabedian & Korn [3]), or using panel methods based on source, doublet and vortex boundary elements (such as Hess & Smith [7], Hunt [8], Katz & Plotkin [10] and Mateescu [12]). More recently, computational solutions have been obtained using various numerical methods for solving the Euler or Navier-Stokes equations, such as those based on finite-difference or finite-volume formulations (for example see Anderson [2] and Mateescu & Stanescu [15]).

The theoretical solutions, and especially those obtained in closed form (such as [14]), are obviously characterized by an excellent computational efficiency in comparison with those requiring intensive numerical calculations. However, only linear theoretical solutions are presently available for the general aerofoil and wing shapes used in the aerospace applications, having thus a limited accuracy. The present work is a part of a more general effort aiming to develop efficient non-linear theoretical solutions in closed form for aerofoils and wings of general shapes in steady and unsteady flows, capable of including also the viscous effects.

The method of velocity singularities, first introduced for the study of wings and wing-body systems in supersonic flow [5, 11], and then extended for the study of aerofoils in subsonic flow [12-14], has proven very efficient in deriving closed form solutions for steady flow problems. This paper presents several methods of solutions developed for steady and unsteady incompressible flows using velocity singularities: (a) analysis of unsteady flows past oscillating aerofoils; (b) analysis of steady flows past wings of finite span; (c) non-linear analysis of steady flows past aerofoils. Efficient solutions in closed form have been obtained in all cases.

2 Unsteady flows past oscillating aerofoils

The method of solution for this problem is illustrated here for the case of an oscillating thin aerofoil of chord \( c \) executing harmonic oscillations about its mean position situated along the axis \( Ox \), which are defined by the equation

\[
y = e(x, t) = \hat{e}(x) \exp(i \omega t), \quad (i = (-1)^{1/2})
\]

where \( x \) and \( y \) represent the Cartesian coordinates with the origin \( O \) placed at the aerofoil leading edge in its mean position, \( t \) is the time, \( \omega \) is the radian frequency of oscillations, and \( \hat{e}(x) \) defines the modal amplitude of oscillation in flexure or as a rigid plate. In the case of rigid thin aerofoil oscillations in translation, 

\[
h(t) = \hat{h} \exp(i \omega t),
\]

and pitching rotation,

\[
\theta(t) = \hat{\theta} \exp(i \omega t),
\]

the modal amplitude is

\[
\hat{e}(x) = \hat{h} + x \hat{\theta}.
\]

In the complex form convention used above to define the oscillations, the reduced quantities marked by a hat, \(^\wedge\), are complex numbers.

The oscillating aerofoil is placed in an undisturbed uniform flow of velocity \( U_\infty \) parallel to the \( x \)-axis, and the unsteady perturbation velocity components generated by the aerofoil oscillations are denoted by \( u \) and \( v \).

The boundary condition on the oscillating aerofoil can be expressed as

\[
v = V(x, t) = \hat{V}(x) \exp(i \omega t), \quad \hat{V}(x) = i \omega \hat{e}(x) + (U_\infty + u) \frac{\partial \hat{e}}{\partial x}.
\]
Consider the reduced perturbation velocity components \( \hat{u} = u / \exp(\ii \omega t) \) and \( \hat{v} = v / \exp(\ii \omega t) \). Since the velocity components are harmonic functions, satisfying the Laplace equation, one can introduce the complex conjugate reduced velocity \( \hat{w} = \hat{u} - \ii j \hat{v} \), where \( j = (-1)^{1/2} \), and then the boundary condition on the oscillating aerofoil can be expressed in the complex form

\[
\text{IMAG}(\hat{w})_{zz} = -\hat{V}(x) \quad \text{for} \quad 0 < z = x < c .
\]  

(3)

Denote by \( \hat{\Gamma}(c) = \Gamma(c,t) / \exp(\ii \omega t) \) the reduced circulation around the aerofoil. By using Kelvin's circulation theorem [4, 17, 19], the intensity of the shedding free vortices at the trailing edge, \( d\hat{\Gamma}(c,t) = d\hat{\Gamma}(c) \exp(\ii \omega t) \) can be derived as

\[
d\hat{\Gamma}(c) = \hat{\gamma}_f (c) U_\infty \, dt , \quad \text{where} \quad \hat{\gamma}_f (c) = -(i \omega / U_\infty) \hat{\Gamma}(c) .
\]  

(4)

Since the shedding free vortices are moving downstream with the fluid velocity, a free vortex situated at \( x = \sigma \) at the time \( t \) was initiated at the trailing edge at a previous time, \( t - \Delta t \), where the time lag is \( \Delta t = (\sigma - c) / U_\infty \). As a result, the intensity of the free vortex distribution at \( x = \sigma \) is

\[
\hat{\gamma}_f (x) = 2 \hat{U}(\sigma) , \quad \hat{U}(\sigma) = -(i \lambda / c) \hat{\Gamma}(c) \exp(-i2\lambda[\sigma - c]/c) ,
\]  

(5)

where \( \lambda = \omega c / (2 U_\infty) \) is the reduced frequency. Thus, the boundary condition on the aerofoil wake may be expressed in the complex form (since \( \hat{u} = \hat{\gamma}_f / 2 \))

\[
\text{REAL}(\hat{w})_{zz} = \hat{U}(x) \quad \text{for} \quad z = x > c .
\]  

(6)

Consider the prototype problem defined by the boundary conditions

\[
\text{IMAG}\{\delta \hat{w}\}_{z=\sigma} = \begin{cases} b_0 & \text{for} \ x \in (0, s) \\ -\delta \hat{V} & \text{for} \ x \in (s, c) \end{cases} , \quad \text{REAL}\{\delta \hat{w}\}_{z=\sigma} = \begin{cases} 0 & \text{for} \ x < 0 \\ \hat{U}(x) & \text{for} \ x > c \end{cases} .
\]  

(7)

As shown in our previous papers [12-14], the following velocity singularities are associated with the leading edge \( (x=0) \) and with the ridges \( x=s \) and \( x=\sigma \) where the imaginary and real parts, respectively, have sudden changes:

\[
\sqrt{\frac{c-z}{z}} - \frac{2}{\pi} \delta \hat{V} \cosh^{-1}\left(\frac{(c-z)s}{c(s-z)}\right) - \frac{2}{\pi} \left(\frac{d\hat{U}}{dz}\right)_{z=\sigma} \cos^{-1}\left(\frac{(c-z)\sigma}{c(\sigma-z)}\right).
\]  

(8)

Thus, the solution of the prototype problem for \( y > 0 \) is

\[
\delta \hat{W}(z) = -j b_0 + \delta A \sqrt{\frac{c-z}{z}} - \frac{2}{\pi} \delta \hat{V} \cosh^{-1}\left(\frac{(c-z)s}{c(s-z)}\right) + \frac{2}{\pi} \lim_{\sigma^+ \to \infty} \int_0^\sigma \frac{d\hat{U}}{d\sigma} \cos^{-1}\left(\frac{(c-z)\sigma}{c(\sigma-z)}\right) d\sigma .
\]  

(9)

The constant \( \delta A \) is obtained from the condition \( \{\delta \hat{W}(z)\}_{z=\sigma} = 0 \), and the reduced circulation around the aerofoil is defined as \( \hat{\Gamma}(c) = 2 \int_0^c \text{REAL}[\delta \hat{W}(z)]_{z=x} dx \). The reduced pressure coefficient \( \delta \hat{C}_p = -\frac{2}{U_\infty} \left[ \frac{i \lambda}{c} \hat{\Gamma}(x) + \text{REAL}[\delta \hat{W}(x)] \right] \) becomes
\[
U - \frac{1}{2} \delta \hat{C}_p(x) = \sqrt{\frac{c - x}{x}} \left[ 2i\lambda \frac{x}{c} + C(\lambda) \right] b_0 + \frac{2}{\pi} \delta \hat{V} \cos^{-1} \left( \frac{s}{c} \right) \\
+ \frac{2}{\pi} \delta \hat{V} \cosh^{-1} \left( \frac{c - x}{c} \frac{s}{s + c} \right) \left[ 2i\lambda \frac{x - s}{c} + 1 \right] - \frac{2}{\pi} \delta \hat{V} \left[ 1 - C(\lambda) \right] \sqrt{\frac{c - x}{x}} \left( \frac{s}{c} \right)
\]

where \( C(\lambda) = H_1^{(2)}(\lambda) / \left[ H_1^{(2)}(\lambda) + iH_0^{(2)}(\lambda) \right] \) is Theodorsen’s function [16, 22].

For the complete unsteady flow problem of an oscillating aerofoil, defined by the boundary condition (2), the solution can be obtained by introducing \( \delta \hat{V} = [d\hat{V}(x)/dx]_s ds \) in equation (10) and then integrating with respect to \( s \).

In the case of oscillations in translation and rotation, as well as of flexural oscillations, the reduced boundary condition (2) can be expressed in the general polynomial form,

\[
\hat{V}(x) = U \sum_{k=0}^{n} b_k x^k,
\]

which leads to the following expression of the reduced pressure coefficient on the upper surface of the aerofoil [16]:

\[
\hat{C}_p(x) = 2 \sqrt{\frac{c - x}{x}} \sum_{j=0}^{n} A_j x^j,
\]

\[
A_j = \sum_{k=0}^{n} b_k \left[ g_{k-j} \left( 1 - \delta_{j0} \left[ 1 - C(\lambda) \right] \frac{2k + 1}{k + 1} \right) + \frac{2i\lambda}{c} g_{k-j+1} \left( 1 + \delta_{j1} \right) \right] + \delta_{j0} b_0 g_0 \frac{2i\lambda}{c},
\]

where \( \delta_{jl} = 1 \) for \( j = l \) and \( \delta_{jl} = 0 \) for \( j \neq l \).

Thus, a remarkably simple expression in closed form was obtained for the unsteady pressure coefficient, \( \hat{C}_p(x, t) = \hat{C}_p(x) \exp(i\omega t) \), on the aerofoil, in the general case of oscillations in translation and rotation or flexural oscillations.

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**Figure 1:** Typical variation in time of the unsteady lift coefficient \( K_L(t) \) for pitching oscillations (—— present solution; o Theodorsen’s results).
By integrating the pressure coefficient along the aerofoil one can also obtain simple solutions for the unsteady lift and pitching moment coefficients. As an example, the unsteady lift coefficient is

\[ C_L(t) = \sum_{k=0}^{\infty} \left[ \frac{i\lambda}{k+2} + 2C(\lambda) \right] \frac{2k+1}{k+1} b_k g_k \]  

(12)

The present method of solution was validated by comparison with Theodorsen's solutions obtained for the case of the rigid aerofoil oscillations in translation and pitching rotation. An excellent agreement was found between the present solutions and Theodorsen's results [16]. This is illustrated for the case of pitching oscillations, \( \dot{V}(x) = -U_{\infty}\dot{\theta} - i\omega\dot{\theta} x \) (or \( b_0 = -\dot{\theta}, b_1 = -2i\lambda\dot{\theta} c \), and \( b_k = 0 \) for \( k > 1 \)) in Figure 1, which shows the typical variation in time of the non-dimensional unsteady lift coefficient \( K_L(t) = \text{REAL}\{\hat{K}_L \exp(\imath\omega t)\} \), where \( \hat{K}_L = \hat{C}_L / (\pi \theta_0 \lambda^2) \) for \( \lambda = 0.34 \) and \( \lambda = 0.24 \).

3 Aerodynamics of wings of finite span

The method of solution for this problem is illustrated here for the cases of the steady flow past a thin wing of span \( 2b \) and variable chord \( c(y) \), placed in a uniform flow of velocity \( U_{\infty} \). Figure 2 shows a trapezoidal wing of taper ratio \( q = 1 - c_e/c_0 \), where \( c_0 \) is the central chord and \( c_e \) is the chord at the wing extremity. The fluid flow is referred to a Cartesian reference system \( b_x, b_y, b_z \), where \( x, y \) and \( z \) are non-dimensional coordinates, with \( y \) along the wing span. In general, the wing incidence with respect to \( U_{\infty} \) (including also the camber effect) may vary along the span, i.e. \( \alpha = \alpha(y) \). Let \( u, v, w \) denote the perturbation velocity components around the wing, and \( v^*, w^* \) denote the corresponding perturbation velocity components in the Trefftz plane situated at \( x = \infty \). All the above velocity components are harmonic functions, satisfying the Laplace equation. As a result, a complex conjugate velocity, \( U(X) = v^*(y, z) - i w^*(y, z) \), may be defined in the Trefftz plane, where \( X = y + i z \) is a complex variable. Consider in this Trefftz plane the prototype problem defined by the boundary conditions.
As shown in our previous papers [11-14, 5], the following velocity singularities are associated with the wing edges \((bX = \pm b)\) and ridges \(bX = \pm b s\) (points where the imaginary part has sudden changes), respectively:

\[
\frac{1}{\sqrt{1-X^2}} + \frac{2}{\pi} \delta W \cosh^{-1} \frac{(1-X)(1+s)}{2(s+X)}.
\]

Thus, the solution of this prototype problem for \(z > 0\) in the Trefftz plane is [18]

\[
\delta U(X) = i\beta_0 + \delta A \frac{X}{\sqrt{1-X^2}} - \delta W \left[ G_1(1,s,X) - G_2(1,s,X) \right],
\]

where the constant \(\delta A\) is obtained from the condition \(\left[ \delta U(X) \right]_{X=\infty} = 0\):

\[
\delta A = \beta_0 + \delta W(4/\pi) C(1,s), \quad \text{where} \quad C(1,s) = \cos^{-1} \sqrt{(1+s)/2}.
\]

The elementary circulation on the wing trace in the Trefftz plane is

\[
d[\delta \Gamma(y)] = -2 \text{REAL} \left[ \delta U(X) \right]_{X=\infty} dy,
\]

resulting after integration

\[
\delta \Gamma(y) = 2 \delta A \sqrt{1-y^2} - \delta W \left[ (y-s)G_1(1,s,y) + \sqrt{1-s^2} C(1,y) \right] + \delta W \left[ (y+s)G_2(1,s,y) + \sqrt{1-s^2} C(1,y) \right],
\]

Consider in the Trefftz plane a general polynomial representation for \(w^*(y)\), corresponding to a symmetrical distribution of the wing incidence with respect to the central chord, in the form

\[
w^*(y) = \sum_{k=0}^{n} \beta_{2k} y^{2k},
\]

where the coefficients \(\beta_{2k}\) are not yet specified. In this case, introducing the vertical velocity change

\[
\delta W = \pm [dW(y)/dy]_{y=x} dy \quad \text{in equation} \ (17), \quad \text{one obtains after integration with respect to} \ s, \ [18],
\]

\[
\Gamma(y) = 2 \sqrt{1-y^2} \left[ \beta_0 + \sum_{k=1}^{n} \beta_{2k} \frac{1}{2k+1} \sum_{j=0}^{k} I_{2(j-1)} y^{2j} \right],
\]

where:

\[
I_j = \left[ (j-1)/j \right] I_{j-2}, \quad I_0 = 1, \quad I_1 = 2/\pi.
\]

This is indeed a very simple solution in closed form, especially in comparison with the previous solutions obtained by Fourier expansions for this problem.

In the lifting line theory [4], the spanwise variation of the circulation around the wing can be expressed as
\[
\Gamma(y) = K \c(y)[U_\infty \alpha(y) - w(y)] , \quad w(y) = \frac{w^*(y)}{2} = \frac{1}{2} \sum_{k=0}^{n} \beta_{2k} y^{2k} , \quad (19)
\]

where \( K = (0.85 - 0.95) \pi \) (to include the viscous effects on the lift coefficient slope), \( c(y) \) and \( \alpha(y) \) are the spanwise variations of the wing chord and incidence, and \( w(y) \) represents the velocity induced on the wing lifting line by the semi-infinite free vortex sheet, which is equal to half of the velocity \( w^*(y) \) induced in the Trefftz plane by the infinite free vortex sheet [4, 18].

The above lifting line solution is used only to determine the coefficients \( \beta_{2k} \), by imposing the expressions (18) and (19) of \( \Gamma(y) \) to be equal at \( n + 1 \) collocation points along the wing semi-span.

The theoretical solutions obtained with the present velocity singularity method have been found in excellent agreement with the results obtained by Carafoli for flat wings at constant incidence [4, 18], as shown in Figure 3 for the case of a rectangular wing \( (q = 0, \alpha(y) = \alpha_0, \text{aspect ratio } \lambda = 5.65, N = 10) \).

![Figure 3](typical_spanwise_variation.png)

Figure 3: Typical spanwise variation of the non-dimensional circulation, \( \Gamma(y)/bU_\infty \alpha_0 \) ( —— present solution; o Carafoli’s solution).

4 Non-linear solutions for aerofoils of symmetrical thickness

The non-linear method of solution developed for relatively thick aerofoils is illustrated here for the case of the steady flow past an aerofoil of symmetrical thickness, which is situated in a uniform stream of velocity \( U_\infty \) parallel to its chord. The flow is referred to a Cartesian reference system \( cx \) and \( cy \) with the origin at the leading edge, where \( x \) and \( y \) are non-dimensional coordinates. The relative semi-thickness of the aerofoil, \( e(x) = e^*(x)/c \), where \( c \) is the chord
length and $e^*(x)$ is half of the dimensional aerofoil thickness, is defined by the
following general polynomial expression, which is suitable to represent the
aerofoils used in the aeronautical applications, such as NACA aerofoils,

$$e(x) = e_0 \sqrt{x} + \sum_{k=1}^{n} e_k x^k.$$  \hspace{1cm} (20)

The non-linear boundary condition can be expressed for the upper surface of the
aerofoil in the form $v(X) = \left[1 + u(x) \right] e^*(x)$, where $u = U^* / U$ and $v = v^* / U$ are the
non-dimensional perturbation velocity components, and $e^*(x)$ is the local
slope of the aerofoil. The velocity components are harmonic functions, satisfying
the Laplace equation and thus one can introduce the complex conjugate reduced
velocity $w(z) = u(x, y) - i v(x, y)$, where $z = x + i y$. Since the non-linear
boundary condition on the aerofoil contains the unknown velocity component $u$,
we will consider for it the following general expression on the aerofoil in the form

$$u(x) = -1 + \sqrt{1-x} x \sum_{j=0}^{m} a_j x^j,$$  \hspace{1cm} (22)

which satisfies the leading and trailing edges as stagnation points, and in which
the unknown coefficients $a_j$ will be determined after the nonlinear solution will be
derived. The nonlinear boundary conditions can be expressed in complex form as

$$\text{IMAG}\{w\}_{x=0} = -V(x) \quad \text{for} \quad x \in (0,1), \quad \text{IMAG}\{w\}_{x=\pm l} = 0 \quad \text{for} \quad x < 0, \ x > 1,$$

where

$$V(x) = \sqrt{1-x} x \sum_{j=0}^{m} a_j x^j \left[ \frac{e_0}{2 \sqrt{x}} + \sum_{k=1}^{n} k e_k x^{k-1} \right].$$  \hspace{1cm} (23)

In this case, the singularities associated with both the aerofoil edges and ridges
($x = s$) are of logarithmic type, $\log (x - s)$, which leads to the complex solution

$$w(z) = -\frac{1}{\pi} \int_{s-x}^{s} V(s) \frac{1}{s-x} \ ds,$$  \hspace{1cm} (24)

After performing the integration and taking the real part, \cite{17}, one obtains the
non-linear solution for $u(x)$ on the upper side of the aerofoil in the form

$$u(x) = -\sum_{j=0}^{m} a_j \left\{ e_0 \left[ x^j \sqrt{1-x} \ \sinh^{-1}\sqrt{1-x} / x - \frac{1}{\pi} L_j + \frac{1-x}{\pi} \sum_{q=0}^{j-1} L_{j-1-q} x^q \right] ight. $$

$$+ \left. \sum_{k=1}^{n} k e_k \left[ \left( \frac{1}{2} - x \right) x^{k-1} + \sum_{q=0}^{k-2-j} \left( g_{k-1+j-q} - g_{k+j-q} \right) x^q \right] \right\},$$  \hspace{1cm} (25)

where $L_0 = 1, \ L_j = \frac{2 j}{2 j+1} L_{j-1}$, $g_0 = 1$, $g_j = \frac{2 j-1}{2 j} g_{j-1}$, \hspace{1cm} (26)

The unknown coefficients $a_j$ in this non-linear solution can now be determined
by imposing the non-linear solution (25) to be equal with the assumed expression
(22) at $m+1$ collocation points along the aerofoil.

The non-linear solution derived above is illustrated in Figure 4 for a NACA
0009 aerofoil, characterized by a maximum relative thickness of 9%. The present
non-linear solution in closed form was found in very good agreement with the
experimental results [1] as shown in Figure 4; by contrast, the agreement of the linear solution is not good, especially towards the aerofoil extremities.

![Figure 4: Chordwise variation of the pressure coefficient for NACA 0009 aerofoil of symmetrical thickness, $C_p = 1 - (1 + u)^2/[1 + e^{12}]$ (--- present non-linear solution; o experimental results).](image)

5 Conclusions

Several methods of solutions based on velocity singularities have been developed for the analysis of steady and unsteady flows past fixed and oscillating aerofoils and wings. All these methods led to very efficient theoretical solutions in closed form, which distinguishes them in comparison with the previous solutions.

The solutions obtained for the unsteady flows past oscillating aerofoils have been validated in comparison with Theodorsen's results for oscillations in translation and pitching rotation, displaying an excellent agreement in all cases. In addition to the rigid aerofoil oscillations studied by Theodorsen, the present method led to efficient theoretical solutions in closed form also for the case of flexural oscillations of flexible aerofoils, fitted or not with oscillating ailerons, which is of interest for the aeroelastic studies in the aeronautical applications.

The theoretical solutions of the flow past wings of finite span have been obtained by using velocity singularities in the Trefftz plan. They have been found in excellent agreement with Carafoli's solutions obtained for flat trapezoidal wings of constant incidence. In addition, the present method permitted to obtain easily solutions for more general problems of practical interest, such as twisted wings of variable incidence, wings with symmetrically rotated flaps and anti-symmetrically rotated ailerons, wings executing rolling rotations, etc.
The non-linear method of solution developed based on velocity singularities for the analysis of the flows past aerofoils of symmetrical thickness led also to efficient non-linear solutions in closed form, which were found in good agreement with the experimental results for NACA aerofoils; to the author’s best knowledge these are the first non-linear theoretical solutions in closed form obtained.

References