A Green's function-based iterative approach to the pricing of American options

M. Y. Melnikov
Labry School of Business and Economics, Cumberland University, USA

Abstract

An iterative semi-analytic procedure is developed for solution of problems arising in the pricing of American options. Introduction of a penalty function reduces the problem to a European options problem with a nonlinear term in the Black-Scholes equation. The approach is based on the use of a Green's function constructed for a terminal-boundary value problem stated for the linear Black-Scholes equation. Different boundary conditions can potentially be treated within this approach. Closed analytic form of Green's functions are obtained by a combination of the methods of Laplace transform and variation of parameters. A numerical experiment reveals high accuracy level attained in computing of solutions of linear problems that arise at each stage of the iterative procedure.
Keywords: American options, Green's function, iterative approach.

1 Introduction

Upon implementing a penalty function approach in the way suggested in [1] and later used in [2], the pricing of American options can be simulated by the following nonlinear terminal-boundary value problem

$$\frac{\partial v(S,t)}{\partial t} + \frac{\sigma^2 S^2}{2}\frac{\partial^2 v(S,t)}{\partial S^2} + rS\frac{\partial v(S,t)}{\partial S} - rv(S,t) + \frac{\varepsilon C}{v(S,t)+\varepsilon - q(S)} = 0 \tag{1}$$

$$v(S,T) = \varphi(S) \tag{2}$$

$$v(S_1,t) = A(t) \quad \text{and} \quad v(S_2,t) = B(t) \tag{3}$$

posed for the two variable function $v(S,t)$ in the rectangular region $\Omega = (S_1 < S < S_2) \times (0 < t < T)$ of the S, t-plane.

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In the above setting, $v = v(S,t)$ is the price of the derivative product, $\varphi(S)$ is the pay-off function of a given derivative problem at the expiration time T, with S and t being the share price of the underlying asset and time, respectively. The parameters σ, and r represent the volatility of the underlying asset and the risk-free interest rate, while $q(S)$ represents a barrier function. Here, $0 < \varepsilon \ll 1$ and C are a small regularization parameter and is a positive constant, respectively. Note that smoothness is not required for the functions $\varphi(S)$, $A(t)$ and $B(t)$ that specify the terminal and boundary conditions.

By the evident substitution

$$v(S,t) = V(S,t) + \frac{(S_2 - S)A(t) + (S - S_1)B(t)}{S_2 - S_1} \tag{4}$$

the terminal-boundary value problem in (1)–(3) reduces to

$$\frac{\partial V(S,t)}{\partial t} + \frac{\sigma^2 S^2}{2}\frac{\partial^2 V(S,t)}{\partial S^2} + rS\frac{\partial V(S,t)}{\partial S} - rV(S,t) = F(V(S,t),S,t) \tag{5}$$

$$V(S,T) = f(S) \tag{6}$$

$$V(S_1,t) = 0 \quad \text{and} \quad V(S_2,t) = 0 \tag{7}$$

with the homogeneous boundary conditions imposed at $S = S_1$ and $S = S_2$. The right-hand side function $F(V(S,t),S,t)$ in the governing equation (5) is determined by the substitution in (4) as

$$\begin{aligned} F(V(S,t),S,t) = {} & -\varepsilon C(S_2 - S_1)/\{(S_2 - S_1)[V(S,t) + \varepsilon - q(S)] \\ & + (S_2 - S)A(t) + (S - S_1)B(t)\} + \frac{rS}{S_2 - S_1}[A(t) - B(t)] \\ & - \left(\frac{d}{dt} - r\right)\left(\frac{S_2 - S}{S_2 - S_1}A(t) + \frac{S - S_1}{S_2 - S_1}B(t)\right) \end{aligned} \tag{8}$$

while the function $f(S)$ in (6) is defined by the substitution in (4) in terms of the pay-off function $\varphi(S)$ as

$$f(S) = \varphi(S) - \frac{(S_2 - S)A(T) + (S - S_1)B(T)}{S_2 - S_1} \tag{9}$$

A semi-analytic approach, which is proposed in this study to the problem in (5)–(7), is based on the Green's function method. In [3] one can find an effective algorithm developed for the construction of closed analytic representations of Green's functions to a variety of terminal-boundary value problems stated for the Black-Scholes equation. Earlier in [4], we had also investigated the potential of the Green's function method in solving linear terminal-boundary value problems for the Black-Scholes equation.

To make this presentation complete, we construct, in what follows, the Green's function, which is later employed for the solution of the problem in (5)–(7).

2 Construction of Green's function

Consider the terminal-boundary value problem in (6) and (7) stated for the homogeneous Black-Scholes equation

$$\frac{\partial V(S,t)}{\partial t}+\frac{\sigma^2 S^2}{2}\frac{\partial^2 V(S,t)}{\partial S^2}+rS\frac{\partial V(S,t)}{\partial S}-rV(S,t)=0 \tag{10}$$

in Ω. By implementing the standard substitutions

$$x=\ln S \quad \text{and} \quad \tau=\frac{\sigma^2}{2}(T-t) \tag{11}$$

and setting $u(x,\tau)=V(S,t)$, the terminal-boundary value problem in (10), (6) and (7) converts to the following initial-boundary value problem in $u(x,\tau)$

$$\frac{\partial u(x,\tau)}{\partial \tau}=\frac{\partial^2 u(x,\tau)}{\partial x^2}+(c-1)\frac{\partial u(x,\tau)}{\partial x}-cu(x,\tau) \tag{12}$$

$$u(x,0)=f(\exp x) \tag{13}$$

$$u(a,\tau)=0 \quad \text{and} \quad u(b,\tau)=0 \tag{14}$$

stated on the rectangle $\widetilde{\Omega}=(a<x<b)\times(0<\tau<T)$, on which the region Ω maps by the change of variables introduced in (11). The end-values a and b of the variable x are evidently determined in terms of S_1 and S_2 as

$$a=\ln S_1 \quad \text{and} \quad b=\ln S_2$$

while the parameter c is defined in terms of the coefficients of the Black-Scholes equation as $c=2r/\sigma^2$.

Applying the integral Laplace transform

$$U(x;s)=\int_0^\infty \exp(-s\tau)u(x,\tau)d\tau$$

to the setting in (12)–(14), one arrives at the following boundary value problem

$$\frac{d^2U(x;s)}{dx^2}+(c-1)\frac{dU(x;s)}{dx}-(s+c)U(x;s)=-f(\exp x) \tag{15}$$

$$U(a;s)=0, \quad U(b;s)=0 \tag{16}$$

for the transform $U(x;s)$.

In compliance with the method of variation of parameters, the general solution to (15) is found in the form

$$U(x,s)=\int_a^x \frac{\exp\alpha(x-\xi)}{2\omega}[\exp\omega(\xi-x)-\exp\omega(x-\xi)]f(\exp\xi)d\xi \\ +Q(s)\exp(\alpha+\omega)x+R(s)\exp(\alpha-\omega)x \tag{17}$$

where the parameter ω is introduced as $\omega = (s+\beta)^{1/2}$, while the parameters α and β are defined in terms of c as

$$\alpha = \frac{1-c}{2} \quad \text{and} \quad \beta = \left(\frac{1+c}{2}\right)^2 \tag{18}$$

Satisfaction of the boundary conditions at $x = a$ and $x = b$ yields the system of linear algebraic equations

$$\begin{pmatrix} \exp(\alpha+\omega)a & \exp(\alpha-\omega)a \\ \exp(\alpha+\omega)b & \exp(\alpha-\omega)b \end{pmatrix} \begin{pmatrix} Q(s) \\ R(s) \end{pmatrix} = \begin{pmatrix} 0 \\ \Psi(s) \end{pmatrix} \tag{19}$$

in $Q(s)$ and $R(s)$. Here

$$\Psi(s) = -\int_a^b \frac{1}{2\omega}[\exp(\alpha-\omega)(b-\xi) - \exp(\alpha+\omega)(b-\xi)]f(\exp\xi)d\xi$$

Solving the system in (19), we obtain the functions $Q(s)$ and $R(s)$ as

$$Q(s) = \int_a^b \frac{\exp(\alpha-\omega)a\exp\alpha(b-\xi)}{2\omega[\exp\omega(a-b) - \exp\omega(b-a)]} \times [\exp\omega(\xi-b) - \exp\omega(b-\xi)]f(\exp\xi)d\xi$$

and

$$R(s) = -\int_a^b \frac{\exp(\alpha+\omega)a\exp\alpha(b-\xi)}{2\omega[\exp\omega(a-b) - \exp\omega(b-a)]} \times [\exp\omega(\xi-b) - \exp\omega(b-\xi)]f(\exp\xi)d\xi$$

Upon substituting these in (17), the latter reads as

$$\begin{aligned} U(x,s) = &\int_a^x \frac{\exp\alpha(x-\xi)}{2\omega}[\exp\omega(\xi-x) - \exp\omega(x-\xi)]f(\exp\xi)d\xi \\ &+ \int_a^b \frac{\exp\alpha(x-\xi)[\exp\omega(x-a) - \exp\omega(a-x)]}{2\omega[\exp\omega(a-b) - \exp\omega(b-a)]} \\ &\times [\exp\omega(\xi-b) - \exp\omega(b-\xi)]f(\exp\xi)d\xi \end{aligned}$$

which can be expressed in a single-integral form as

$$\begin{aligned} U(x;s) = &\int_a^b \frac{\exp\alpha(x-\xi)}{2\omega[\exp\omega(a-b) - \exp\omega(b-a)]} \\ &\times \{\exp\omega[(x+\xi)-(a+b)] + \exp\omega[(a+b)-(x+\xi)] \\ &- \exp\omega[|x-\xi| + (a-b)] - \exp\omega[(b-a) - |x-\xi|]\}f(\exp\xi)d\xi \end{aligned}$$

Transforming the bracket factor in the denominator as

$$\exp\omega(a-b) - \exp\omega(b-a) = -\exp\omega(b-a)[1-\exp 2\omega(a-b)]$$

we rewrite the above representation for $U(x;s)$ as

$$U(x;s) = -\int_a^b \frac{\exp\alpha(x-\xi)}{2\omega\exp\omega(b-a)[1-\exp 2\omega(a-b)]} \times \{\exp\omega[(x+\xi)-(a+b)] + \exp\omega[(a+b)-(x+\xi)] - \exp\omega[|x-\xi|+(a-b)] - \exp\omega[(b-a)-|x-\xi|]\} f(\exp\xi)d\xi \tag{20}$$

An immediate inverse Laplace transform of the above expression for $U(x;s)$ is problematic if $U(x;s)$ is kept in its current form. Therefore, we adjust it first by representing the factor

$$[1-\exp 2\omega(a-b)]^{-1}$$

in the integrand of (20) as the sum of the geometric series

$$\frac{1}{1-\exp 2\omega(a-b)} = \sum_{n=0}^{\infty} \exp 2n\omega(a-b)$$

whose common ratio $\exp 2\omega(a-b)$ represents a negative exponential function (since $a < b$) and is, therefore, less than one, justifying convergence of the above series. This transforms the expression for $U(x;s)$ in (20) to

$$\begin{aligned} U(x;s) = \int_a^b \frac{\exp\alpha(x-\xi)}{2\omega} \sum_{n=0}^{\infty} \{ & \exp\omega[|x-\xi| - 2(n+1)(b-a)] \\ & + \exp\omega[2(n+1)(a-b) - |x-\xi|] \\ & - \exp\omega[2n(a-b) - 2b + (x+\xi)] \\ & - \exp\omega[2n(a-b) + 2a - (x+\xi)]\} f(\exp\xi)d\xi \end{aligned}$$

which allows the inverse Laplace transform in the term-by-term fashion. This yields the solution $u(x,\tau)$ to the initial-boundary value problem in (12)–(14) in the form

$$\begin{aligned} & u(x,\tau) \\ & \quad = L^{-1}\{U(x,s)\} \\ & \quad = \int_a^b \frac{\exp\alpha(x-\xi)\exp(-\beta\tau)}{2\sqrt{\pi\tau}} \sum_{n=0}^{\infty} \left\{ \exp\left(-\frac{[|x-\xi| + 2(n+1)(a-b)]^2}{4\tau} \right) \right. \end{aligned}$$

$$+\exp\left(-\frac{[|x-\xi|-2n(a-b)]^2}{4\tau}\right)-\exp\left(-\frac{[2b-(x+\xi)-2n(a-b)]^2}{4\tau}\right)$$
$$-\exp\left(-\frac{[(x+\xi)-2a-2n(a-b)]^2}{4\tau}\right)\Bigg\}f(\exp\xi)d\xi$$

which converts to a more compact form by rearranging the summation in the above series. This yields

$$u(x,\tau)=\int_a^b\frac{\exp\alpha(x-\xi)\exp(-\beta\tau)}{2\sqrt{\pi\tau}}$$
$$\times\sum_{m=-\infty}^{\infty}\left\{\exp\left(-\frac{[|x-\xi|+2m(a-b)]^2}{4\tau}\right)\right.$$
$$\left.-\exp\left(-\frac{[2b-(x+\xi)-2m(a-b)]^2}{4\tau}\right)\right\}f(\exp\xi)d\xi$$

In compliance with the relations in (11), the solution $V(S,t)$ to the setting in (6), (7) and (10) can be obtained by the backward substitution of the variables x, τ and ξ with S, t and $\widetilde{S}$, respectively. When α and β are replaced, according to (18), with the original parameters r and σ^2 of the Black-Scholes equation, we obtain $V(S,t)$ in the form

$$V(S,t)=\int_{S_1}^{S_2}\frac{1}{\sigma\widetilde{S}\sqrt{2\pi(T-t)}}\exp\left(-\frac{r-\sigma^2/2}{\sigma^2}\ln\frac{S}{\widetilde{S}}-\frac{(r+\sigma^2/2)^2}{2\sigma^2}(T-t)\right)$$
$$\times\sum_{m=-\infty}^{\infty}\left\{\exp\left(-\frac{[\ln(S/\widetilde{S})+2m\ln(S_1/S_2)]^2}{2\sigma^2(T-t)}\right)\right.$$
$$\left.-\exp\left(-\frac{[\ln(S_2^2/S\widetilde{S})-2m\ln(S_1/S_2)]^2}{2\sigma^2(T-t)}\right)\right\}f(\widetilde{S})d\widetilde{S}$$

which can be transformed, by combining the logarithmic components in the series factor. This yields

$$V(S,t)=\int_{S_1}^{S_2}\frac{1}{\sigma\widetilde{S}\sqrt{2\pi(T-t)}}\exp\left(-\frac{r-\sigma^2/2}{\sigma^2}\ln\frac{S}{\widetilde{S}}-\frac{(r+\sigma^2/2)^2}{2\sigma^2}(T-t)\right)$$
$$\times\sum_{m=-\infty}^{\infty}\left\{\exp\left(-\frac{[\ln(SS_1^{2m}/\widetilde{S}S_2^{2m})]^2}{2\sigma^2(T-t)}\right)\right.$$
$$\left.-\exp\left(-\frac{[\ln(S_2^{2(m+1)}/S\widetilde{S}S_1^{2m})]^2}{2\sigma^2(T-t)}\right)\right\}f(\widetilde{S})d\widetilde{S} \tag{21}$$

Thus, the kernel

$$G(S,t;\widetilde{S}) = \frac{1}{\sigma\widetilde{S}\sqrt{2\pi(T-t)}}\exp\left(-\frac{r-\sigma^2/2}{\sigma^2}\ln\frac{S}{\widetilde{S}} - \frac{(r+\sigma^2/2)^2}{2\sigma^2}(T-t)\right)$$
$$\times\sum_{m=-\infty}^{\infty}\left\{\exp\left(-\frac{[\ln(SS_1^{2m}/\widetilde{S}S_2^{2m})]^2}{2\sigma^2(T-t)}\right)\right.$$
$$\left. - \exp\left(-\frac{[\ln(S_2^{2(m+1)}/S\widetilde{S}S_1^{2m})]^2}{2\sigma^2(T-t)}\right)\right\} \tag{22}$$

in the integral form of (21) represents the Green's function to the setting in (6), (7) and (10). The series in this representation converges at a high rate unless the difference $(T-t)$ is significantly small. This implies that, in computing approximate values of $G(S,t;\widetilde{S})$, an accuracy level required for applications can, in most cases, be attained by appropriately truncating the series in (22) to a partial sum.

3 Iterative procedure

A Green's function-based algorithm is proposed for solution of the nonlinear terminal-boundary value problem stated in (5)–(7). At the first stage in the algorithm, the function $V(S,t)$ is broken onto two components

$$V(S,t) = V_1(S,t) + V_2(S,t) \tag{23}$$

where $V_1(S,t)$ represents solution of the following linear problem for the Black-Scholes equation

$$\frac{\partial V_1(S,t)}{\partial t} + \frac{\sigma^2 S^2}{2}\frac{\partial^2 V_1(S,t)}{\partial S^2} + rS\frac{\partial V_1(S,t)}{\partial S} - rV_1(S,t) = 0 \tag{24}$$
$$V_1(S,T) = f(S) \tag{25}$$
$$V_1(S,t) = 0 \quad \text{and} \quad V_1(S,t) = 0 \tag{26}$$

while the second component $V_2(S,t)$ in (23) represents solution to the following nonlinear problem

$$\frac{\partial V_2(S,t)}{\partial t} + \frac{\sigma^2 S^2}{2}\frac{\partial^2 V_2(S,t)}{\partial S^2} + rS\frac{\partial V_2(S,t)}{\partial S} - rV_2(S,t) = F(V_2(S,t),S,t)$$
$$V_2(S,T) = 0$$
$$V_2(S_1,t) = 0 \quad \text{and} \quad V_2(S_2,t) = 0$$

As to the linear setting in (24)–(26), we express its solution in terms of the Green's function $G(S,t;\widetilde{S})$ derived in (22) as

$$V_1(S,t) = \int_{S_1}^{S_2} G(S,t;\widetilde{S}) f(\widetilde{S}) d\widetilde{S} \tag{27}$$

Evidently, approximate values of the above integral representation can be computed at any point (S,t) in Ω at any accuracy with the aid of standard quadrature formulae. This study brings a solid justification to this assertion.

The function $V_2(S,t)$ can be computed on an iterative basis. We look for it as the limit of a sequence $\{V_2^{(k)}(S,t)\}$ whose components $V_2^{(k)}(S,t)$, $(k = 0, 1, 2, \ldots)$ are obtained in compliance with the following iterative scheme

$$\frac{\partial V_2^{(k)}(S,t)}{\partial t} + \frac{\sigma^2 S^2}{2} \frac{\partial^2 V_2^{(k)}(S,t)}{\partial S^2} + rS\frac{\partial V_2^{(k)}(S,t)}{\partial S} - rV_2^{(k)}(S,t)$$
$$= F(V_2^{(k-1)}(S,t), S, t), \quad k = 1, 2, 3, \ldots \tag{28}$$

$$V_2^{(k)}(S,T) = 0 \tag{29}$$

$$V_2^{(k)}(S_1,t) = 0 \quad \text{and} \quad V_2^{(k)}(S_2,t) = 0 \tag{30}$$

To run the first loop of the procedure, an appropriate form ought to be chosen for $F(V_2^{(0)}(S,t), S, t)$ in the governing equation of the above setting. The form of $F(0, S, t)$ can, for example, be recommended as an option.

As to the solution $V_2^{(k)}(S,t)$ of the problem in (28)–(30), at each stage ($k = 1, 2, 3, \ldots$) of the iterative procedure, we express it as

$$V_2^{(k)}(S,t) = \int_t^T \int_{S_1}^{S_2} G(S, t-\widetilde{t}, \widetilde{S}) F(V_2^{(k-1)}(\widetilde{S},\widetilde{t}), \widetilde{S}, \widetilde{t}) d\widetilde{S} d\widetilde{t}, \quad k = 1, 2, 3, \ldots \tag{31}$$

in terms of the Green's function $G(S,t;\widetilde{S})$. A standard method of quadratures is also recommended in this case for computing values of $V_2^{(k)}(S,t)$ in Ω.

Convergence of the sequence $\{V_2^{(k)}(S,t)\}$ of successive approximations is yet to be investigated. It depends upon a number of factors one of which is the accuracy level potentially attainable for solutions of those linear problems that arise at each stage of the iterative procedure. The present study focuses on this issue and aims at the creation of a solid reliable background for computing values of the integral representations in (27) and (31). The values of $V_1(S, T/2)$ and $V_2(S, T/2)$ in Tables 1 and 2 were obtained for the following set of initial data: $r = 0.06$, $\sigma = 0.8$, $\varepsilon = 0.1$, $C = 1$, $T = 1$, $S_1 = 1$ and $S_2 = 2$; while the functions $q(S)$, $A(t)$, $B(t)$ and $\varphi(S)$ were defined as

$$q(S) = S, \quad A(t) = 10\exp(-50(t-T/2))^2, \quad B(t) = 20t^3\sin(\pi t/T)$$

and

$$\varphi(S) = \begin{cases} 320(S-S_1)((S_1+S_2)/2 - S), & S < (S_1+S_2)/2 \\ 0, & S > (S_1+S_2)/2 \end{cases}$$

Table 1: Approximate values $V_1(S, T/2)$ attained in computing (27).

Number of quadrature nodes	S coordinate					
	1.0	1.2	1.4	1.6	1.8	2.0
5	.00000	.30506	.44110	.39510	.22473	.00000
10	.00000	.28983	.41908	.37536	.21352	.00000
15	.00000	.29720	.42975	.38492	.21894	.00000
20	.00000	.29445	.42576	.38135	.21692	.00000

Table 2: Approximate values of $V_2(S, T/2)$ attained in computing (31).

Number of quadrature nodes	S coordinate					
	1.0	1.2	1.4	1.6	1.8	2.0
5	.05325	.03736	.02805	.01930	.00986	.00000
10	.01925	.01738	.01627	.01268	.00686	.00000
15	.04806	.03584	.02795	.01967	.01017	.00000
20	.04120	.03107	.02502	.01798	.00937	.00000

The data in Tables 1 and 2 advise that quite accurate values of the integral representations in (27) and (31) can be obtained by using the standard trapezoidal rule with a limited number of quadrature nodes.

4 Conclusion

A semi-analytic approach has been proposed for pricing of American options. It is based on the Green's function method. The approach is supposed to be effective numerically. The following two factors are behind of this assertion. First, the required Green's function is constructed analytically. Second, numerical differentiation, which would substantially deteriorate the accuracy, is completely avoided with only numerical integration (whose accuracy is easy to control) involved.

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