Self-similarity and multifractality in financial asset returns

Ö. Önal

Faculty of Administration and Economics at Marmara University, Turkey

Abstract

This paper presents an empirical investigation of scaling and the multifractal properties of financial asset returns. It discusses the key intuition relative to the applicability of scaling processes and the broader class of multifractal processes to financial phenomena. The cumulative return distribution of positive and negative tails at the different time intervals are linear. This presents strong evidence that returns exhibit power-law scaling in the tails. To test the multifractal properties of returns, we use the sample absolute moment of the aggregated return series. These moments do not scale linearly with different lags. In the other words, the scaling exponent is nonlinear in lags. These results indicate that the returns are multifractal.

Keywords: scaling, self-similarity, multifractality, fat-tails, asset returns, security market.

1 Introduction

Researchers have been investing scaling laws in finance and economics for a long time. The first example of scaling laws in economics is due to the economist Pareto in the 19th century. At the late 1920’s, the work emphasized the appearance of patterns at different time scales. The presence of scaling laws has also been researched in price behaviour. First observed self-similarity in economics time series by Mandelbrot. When he discovered that cotton price time series had approximately the same shape at different time scales [2]. Based on this empirical discovery, Mandelbrot later proposed stable laws and Fractional Brownian Motions as a model for price behaviour. Since Bachelier in 1900 the Wiener process (Brownian Motion) has been used to model asset price returns. This model is the basis, among many other applications, of the Black-Scholes-
Merton option pricing formulae. In spite of its success in various applications, the Wiener process model has come to be widely regarded as an inadequate description of real-life price processes.

A model of asset price returns with the following characteristics would be both theoretically and practically desirable:

- Ability to capture the full extent of observed leptokurtosis (fat-tailedness of the distributions).
- Incorporating non-independence of returns over disjoint time periods, as observed in markets.
- Not a priori excluding the existence of the second moment of the returns process.

The most of the alternatives to the Wiener process model fail to satisfy at least one of these three criteria.

Historically, the thrust of Multifractal theory has been to study the distribution of the fractal dimensions of subsets generated by a deterministic or random mechanism. The multifractal analysis is particularly appropriate in investigating the local fluctuations of these sample paths. The corresponding fractal dimensions measure the “roughness” of the sample paths at localized points within small subsets often of negligible length.

Scaling expresses invariance with respect to translation in time and change in the unit of time [3]. Scaling is a rule that relates returns over different sampling intervals. The shape of the distribution of returns should be the same, when the time scale is changed [1]. Many empirical studies have shown that financial time series exhibit scaling like characteristics, for foreign Exchange rate, we will investigate daily ISE (Istanbul Stock Exchange) index return and Exchange rate return and provide evidence the returns do follow power law. Furthermore, the returns exhibit multifractal behaviour.

The paper is organized as follows. In the next section, we define Self-similar (Monofractal) processes and Multifractals.

In section 3, we indicate how to determine whether a given sample path is better modelled by a Self-similar or Multifractal process. We apply this methodology to index return Exchange rates in section 4 and we present our conclusions in section 5.

2 Self-similarity and multifractality

The concepts of scaling and self-similarity apply to distributions, processes or structures. Self-similarity was introduced as a property that applies to geometrical self-similar objects, i.e., fractal structures. In this context, self-similarity means that a structure can be put into a one to one correspondence with a part of itself.

2.1 Self-similarity

The self-similarity for stochastic processes can be seen as the invariance in distribution under suitable scaling of time scale. Self-similarity entails scaling:
If a fractal structure is expanded by a given factor, its measure expands by a power of the same factor. The notation of scaling is often expressed as absence of scale, meaning that a scaling object looks the same at any scale, large or small [5].

A stochastic process, in fact, is said to have a “scaling property” if there is no natural scale for looking at its paths and distributions. Intuitively, this means that it’s not possible to gauge the scale of a sample by looking at its distribution: There is absence of scale. If a price pattern is generated by a process with scaling property, the plots of average daily and monthly prices will appear to be perfectly similar in distribution: Looking at the plot, it’s impossible to tell if it refers to daily or monthly prices.

Self-similarity is another way of expressing the same concept. A process is self-similar, if a portion of process is similar to the entire process. As we are considering a random environment, self-similarity applies to distributions, not to the actual realization of a process.

A process \( \{X(t), \ t \in \mathbb{R}\} \) is said to be self-similar of index \( H \) (with self-similarity parameter) if, for any \( k > 0 \),

\[
\{X(kt), \ t \in \mathbb{R}\}^d = k^H \{X(t), \ t \in \mathbb{R}\}, \quad 0 < H < 1
\]  

(1)

The canonical example of such a process is Fractional Brownian Motion, Brownian Motion, if \( H = \frac{1}{2} \).

A second definition of self-similarity, more appropriate in the context of standard time series theory, involves a stationary sequence, \( \{X(i), \ i \geq 1\} \), let,

\[X^{(m)}(k) = \frac{1}{m} \sum_{i=(k-1)m+1}^{km} X(i), \quad k = 1, 2, \ldots\]  

(2)

be the corresponding aggregated sequence with level of aggregation \( m \), obtained by dividing the original series \( X \), into non-overlapping blocks of size \( m \) and averaging over each block. The index, \( k \), labels the block. If \( X \) is the increment process of a self-similar process then for all integer \( m \),

\[X \overset{d}{=} m^{1-H} X^{(m)}\]  

(3)

Self-similarity is often investigated not through the equality of finite-dimensional distributions, but through the behaviour of the absolute moments. Thus consider,
\[ \mu^{(m)} (q) = E \left| X^{(m)} \right|^q = E \left| \frac{1}{m} \sum_{i=1}^{m} X(i) \right|^q \]  \hspace{1cm} (4)

If X is self-similar, then \( \mu^{(m)} (q) \) is proportional to \( m^{\beta(q)} \) that is \( \log \mu^{(m)} (q) \) is linear in \( \log m \) for a fixed \( q \):

\[ \log \mu^{(m)} (q) = \beta(q) \log m + c(q) \]  \hspace{1cm} (5)

In addition, the exponent \( \beta(q) \) is linear with respect to \( q \).

In fact, since

\[ X^{(m)} (i) = m^{H-1} X(i), \]  we have,

\[ \beta(q) = q(H-1) \]  \hspace{1cm} (6)

Brownian motion and symmetric \( \alpha \)-Stable Levy Motion are self-similar with stationary increments. In particular, for Brownian motion, it holds that \( H = 1/2 \), while \( H = 1/\alpha \) for Levy motion [4], [6].

### 2.2 Multifractality

Multifractal model of asset returns (MMAR) has been proposed [3]. This model exhibits long tails, although necessarily implying infinite variance of moments and long-range dependence. The main characteristics of MMAR is multiscaling: This property can be regarded as a generalization of self-similarity for stochastic processes.

Multifractality is a form of generalized scaling that includes both extreme variations and long memory [1], while a self-similarity process \( X(t) \) with index \( H \) satisfies the relation;

\[ X(kt) = k^H X(t) \]  \hspace{1cm} (7)

a Multifractal process follows the more general scaling rule;

\[ X(kt) = M(k) X(t) \]  \hspace{1cm} (8)

Where \( M(k) \) is a stochastic processes independent of \( X(t) \).

**Definition:** Let \( X(t) \) be a stochastic process with stationary increments. Let \( T \) and \( Q \) be intervals in \( \mathbb{R} \) such that \( Q \subseteq T \) and \( [0,1] \subseteq Q \). If the moments of \( X(t) \) satisfy the relation,

\[ E \left| X(t)^q \right| = c(q) t^{\tau(q)+1} \]  \hspace{1cm} (9)
for all \( t \in T \), where \( c(\cdot) \) and \( \tau(\cdot) \) are \( \mathbb{R} \) - valued functions defined on \( Q \), then we say that \( X(t) \) is a multifractal process. \( c(q) \) and \( \tau(q) \) are both deterministic functions of \( q \). The function \( \tau(q) \) is called the *scaling function* of the multifractal processes. *Unifractal or uniscaling is a special case of multifractal which has a linear scaling function.* Multifractal processes are characterized by the non-linearity of functions \( \tau(q) \) [4].

### 3 Methodology

If \( P(t) \) is a series of prices, for \( t \in [0, T] \), let, \[
X(t) = \ln P(t) - \ln P(0) \tag{10}
\]
in general, denote the corresponding (instantaneous) returns process. To investigate the multifractal properties of returns, the mean moment of the absolute returns as a function of time intervals for several different values of \( q \) are plotted in a double logarithmic space. To estimate the \( q \)th moment of \( X \), \( \mu^{(m)}(q) \), we use the \( q \)th sample absolute moment of the aggregated series \( X^{(m)} \),

\[
\mu^{(m)}(q) = \frac{1}{N/m} \sum_{k=1}^{N/m} \left| X^{(m)}(k) \right|^q \tag{11}
\]

where \( N \) is sample size.

Our test for multifractality in our time series will consist of computing and plotting \( \ln \mu^{(m)}(q) \) against \( \ln (m) \) for different values of \( q \) and verifying linearity by inspection.

*The return process is monofractal if \( \tau(q) \) is a linear function of \( q \) and multifractal if \( \tau(q) \) is non-linear.* The non-linearity of \( \tau(q) \) is testing by plotting;

\[
E\left(\left| X^{(m)} \right|^q \right) / E\left(\left| X^{(m)} \right| \right)^q \quad \text{versus time intervals for several different values of } q \text{ ranging from 0.5 (bottom) to 4 (top) in a log – log space. The straight lines which are observed have a slope of } \tau(q) - q \tau(1). \text{ In the case that } \tau(q) \text{ is a linear function of } q, \text{ the slopes of the straight lines should be 0. However, if}
\]
\( \tau(q) \) is non linear, a trend should be expected. Then one turns to the series \( X - EX \), that is, 

\[
\hat{\mu}^{(m)}(q) = \frac{1}{N/m} \sum_{k=1}^{N/m} X^{(m)}(k) - \frac{1}{N} \sum_{i=1}^{N} X(i)^q
\]

if \( \ln \hat{\mu}^{(m)}(q) \) scales linearly with \( \ln(m) \), then a multifractal model can apply. If, however, in addition \( \beta(q) \) is linear in \( q \), then a self-similar model is adequate.

The data we analyse are the time series of exchange rate logarithmic variations for the time period 01/01/1998 to 03/11/2003 of the following daily spot foreign currency, American dollar (USD) and daily return series listed on the Istanbul Stock Exchange (ISE) 100 index for the time period 01/01/1998 to 03/11/2003.

To investigate the multifractal properties of USD and ISE 100 index returns, the mean moment of the absolute returns as a function of time intervals for several different value of \( q \) are plotted in a double logarithmic space figure (top) plots \( E\left(|X^{(m)}|^q\right) \) against time intervals for different values of \( q \). Time intervals range from 1 to 30 days. The value of \( q \) increase 0.5 to 3.5: The \( \tau(q) \) is linear function of \( q \) and multifractal, if moments show different slope for different value of \( q \), which suggest different scaling laws for different order of moments. The return process is monofractal, if \( \tau(q) \) is a linear function of \( q \) and multifractal, if \( \tau(q) \) is nonlinear. The variation of the line’s slope with \( q \) is nonlinear, suggestion multifractal behavior of return process.

4 Conclusions

This paper has investigated self-similarity and multifractal properties of USD and ISE 100 index returns. Returns are not self-similar. Instead they follow a multifractal scaling law. The relation of the mean moment of absolute returns and time intervals at different order of moment are examined.

The linear relationship between the moments and time intervals indicate the scaling properties of absolute returns. The non-linearity of the scaling exponent provides for multifractal properties of USD and ISE 100 index returns.

References


