Optimal quasi-Monte Carlo valuation of derivative securities

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Abstract

For many applications of computational finance, the use of quasirandom sequences seems to provide a faster rate of convergence than pseudorandom sequences. However, at present there are only a few choices of quasirandom sequences. By scrambling a quasirandom sequence we can produce a family of related quasirandom sequences. Finding an optimal quasirandom sequence within this family is an interesting problem, as such optimal quasirandom sequences can be quite useful in applications. The process of finding such optimal quasirandom sequences is called the derandomization of a randomized (scrambled) family. In this paper, we summarize aspects of this technique and explore applications of optimal quasirandom sequences to evaluate a particular derivative security. We find that the optimal quasirandom sequences give promising results even for high dimensions.

Keywords: geometric Asian options, derandomization, optimal Faure sequences, scrambled Faure sequences.

1 Introduction

Numerical methods are used for a variety of purposes in modern finance. These includes risk analysis, the valuation of securities, and the stress testing of portfolios. The Monte Carlo approach, which uses random sequences, has proved to be a valuable computational tool in modern finance. However, for many applications in computational finance, the use of quasirandom sequences, called the quasi-Monte Carlo approach, seems to provide a faster rate of convergence than random sequences. Thus, the generation of appropriate high-quality quasirandom sequences is important to the quasi-Monte Carlo approach to many problems in computational finance. Today, we only have a few choices of practical quasiran-
dom sequences. Fortunately, a recent modification of quasi-Monte Carlo, known as randomized quasi-Monte Carlo [1, 2] provides many more choices to quality quasirandom sequences.

Given a quasirandom sequence, randomness can be added to the sequence in order to obtain a randomized family of quasirandom sequences. Scrambling can help us obtain such a randomized family. The purpose of scrambling in quasi-Monte Carlo is twofold: first, it provides a practical method to obtain error estimates for quasi-Monte Carlo based by treating each scrambled sequence as a different and independent random sample from a family of randomly scrambled quasirandom numbers [1]. Secondly, scrambling can improve the quality of quasirandom sequences. Most of the proposed scrambling methods [3, 1, 4, 5] aim at more uniformity in high-dimensional quasirandom sequences, which can be checked via two-dimensional projections. The core of randomized quasi-Monte Carlo is to find fast and effective algorithms to randomize (scramble) quasirandom sequences.

After a stochastic family of quasirandom sequences is produced, it is a natural question, therefore, to ask how to choose an optimal quasirandom sequence from this family. The purpose of this paper is to find an optimal Faure sequence and uses it to valuate Asian options. The process of searching and specifying optimal quasirandom sequences that achieve theoretically and empirically optimal results is an important problem in quasi-Monte Carlo. The process of finding such optimal quasirandom sequences is commonly called “derandomization”. For example, GFau re (Generalized Faure) [2] is a family of scrambled Faure sequences that has been successfully used in computational finance [6, 7], and is a derandomization of the ordinary Faure sequence.

The rest of this paper is organized as follows. First a brief review of the Faure sequence and GFau re are presented in §2. An algorithm for finding the optimal Faure sequence within a scrambled family is presented in §3. An introduction to Asian options is given, and the numerical results on the Asian options are provided in §4. This is followed by the final section, §5, where we briefly provide conclusions and discuss future work.

2 The scrambled Faure sequence

The original construction of quasirandom sequences was related to the van der Corput sequence, which is a one-dimension quasirandom sequence based on digital inversion. This digital inversion method is a central idea behind the construction of many current quasirandom sequences in arbitrary bases and dimensions. Following the construction of the van der Corput sequence, a significant generalization of this method was proposed by Faure [8] to the sequences that now bear his name. Later, Tezuka [2] proposed the generalized Faure sequence, GFau re, which forms a family of randomized Faure sequences.

The Faure sequence is based on the radical inverse function, \( \phi_b(n) \), and a generator matrix, \( C \). Let \( b \geq 2 \) be prime, and \( n = (n_0, n_1, \ldots, n_{m-1})^T \) be an integer vector with its elements the \( b \)-adic expansion of the integer \( n \). Then the radical
inverse function, $\phi_b(n)$, is defined as
\[
\phi_b(n) = \frac{n_0}{b} + \frac{n_1}{b^2} + \ldots + \frac{n_{m-1}}{b^m}.
\]

The Faure sequence defines a different generator matrix for each dimension. The generator matrix of the $j$th dimension for an $s$-dimensional Faure sequence is denoted as $C^{(j)} = P^{j-1}$ for $(1 \leq j \leq s)$, where $P$, the Pascal matrix, is defined as follows:
\[
P^{j-1} = \binom{r-1}{k-1} (j-1)^{(r-k)_{k \geq 1, r \geq 1}} \pmod{b}.
\]

Above $k$ is the row index, and $r$ is the column index. Thus let $x_n = (x_n^{(1)}, x_n^{(2)}, \ldots, x_n^{(s)})$ be the $n$th Faure point, then $x_n^{(j)}$, can be represented as follows:
\[
x_n^{(j)} = \phi_b(C^{(j)}n),
\]
and so
\[
(\phi_b(P^0n), \phi_b(P^1n), \ldots, \phi_b(P^{s-1}n))
\]
gives the $s$-dimensional Faure sequence.

Tezuka’s G\texttt{Faure} has the $j$th dimension generator matrix as $C^{(j)} = A^{(j)}P^{j-1}$, where $A^{(j)}$ is a random nonsingular lower triangular matrix and can be expressed as follows:
\[
A^{(j)} = \begin{pmatrix}
h_{11} & 0 & 0 & 0 & \ldots & 0 \\
g_{21} & h_{22} & 0 & 0 & \ldots & 0 \\
g_{31} & g_{32} & h_{33} & 0 & \ldots & 0 \\
\vdots & \vdots & \vdots & \ddots & \ddots & \ddots \\
\vdots & \vdots & \vdots & \ldots & \ddots & \ddots \\
\end{pmatrix}_{m \times m},
\]
where \( h_{ii} \) is uniformly distributed on the set \{1, 2, ..., \( b - 1 \)\}, \( g_{ij} \) is uniform on the set \{0, 1, 2, ..., \( b - 1 \)\}, and \( m \) is the number of digits to be scrambled. Thus \( \text{GFau}r \) e is a stochastic family of the Faure sequence, and this family has as many as \( b^{m(m-1)/2}(b - 1)^m \) different sequences. An interesting problem is finding an optimal Faure sequence within such a large family.

I-binomial Scrambling [9] is an algorithm to reduce the number of sequences in this \( \text{GFau}r \) e family while maintaining the original quality of Faure sequence. A subset of \( \text{GFau}r \) e is called “\( \text{GFau}r \) e with the \( i \)-binomial property” [9], with \( A^{(j)} \) defined to be Toeplitz:

\[
A^{(j)} = \begin{pmatrix}
    h_1 & 0 & 0 & 0 & \ldots & 0 \\
    g_2 & h_1 & 0 & 0 & \ldots & 0 \\
    g_3 & g_2 & h_1 & 0 & \ldots & 0 \\
    g_4 & g_3 & g_2 & h_1 & \ldots & 0 \\
    \vdots & \vdots & \vdots & \vdots & \ddots & \ddots \\
    \vdots & \vdots & \vdots & \vdots & \ddots & \ddots \\
\end{pmatrix}_{m \times m},
\]

where \( h_1 \) is uniformly distributed on the set \{1, 2, ..., \( b - 1 \)\}, and \( g_i, 2 \leq i \leq m \), is uniformly on the set \{0, 1, 2, ..., \( b - 1 \)\}. For each \( A^{(j)} \), there will be a different random matrix in the above form.

I-binomial scrambling reduces the scrambling space from \( O(m^2) \) to \( O(m) \). This reduction makes searching for the optimal Faure sequence computationally tractable. Following the lead of i-binomial scrambling proposed by Tezuka and Faure [9], we try to find an optimal Faure sequence from a relatively smaller space, rather than the whole \( \text{GFau}r \) e.

### 3 The optimal Faure sequence

In this section, we provide a number theoretic criterion to choose an optimal scrambling from among a large family of possible (random) scramblings of the Faure sequence. Based on this criterion, we have found the optimal scramblings for any dimension. This derandomized Faure sequence is then numerically tested and shown empirically to be far superior to the original unscrambled sequence.

There have been various scrambling methods proposed for the Faure sequence to obtain better uniformity for quasirandom sequences in high dimensions. Among these scrambling algorithms, the simplest and most effective is linear matrix scrambling [10]. \( \text{GFau}r \) e and i-binomial scrambling are good examples of linear matrix scrambling.

In the rest of this section, our algorithm for searching for an optimal Faure sequence within \( \text{GFau}r \) e with the \( i \)-binomial property is described. The diagonal element, \( h_1 \), of \( A^{(j)} \) in (3) scrambles all digits of each original Faure point. The element \( g_2 \) scrambles all but first digit of that Faure point. Most importantly, the two-most significant digits of the Faure point are only scrambled by \( h_1 \) and \( g_2 \).
Thus the choice of these two elements is crucial for producing optimally scrambled Faure sequences. So we focus on finding the best and simplest values for \( h_1 \) and \( g_2 \) so that an optimal Faure sequence can be obtained. In our example, we consider a simple form for \( A^{(j)} \) within \( i \)-binomial scrambling:

\[
A^{(j)} = \begin{pmatrix}
    h_1^{(j-1)} & 0 & 0 & 0 & \ldots & 0 \\
    g_2 & h_1^{(j-1)} & 0 & 0 & \ldots & 0 \\
    0 & g_2 & h_1^{(j-1)} & 0 & \ldots & 0 \\
    0 & 0 & g_2 & h_1^{(j-1)} & \ldots & 0 \\
    \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
    \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
\end{pmatrix}_{m \times m}.
\]  

(4)

Thus, we search for the optimal Faure sequence in this reduced \( i \)-binomial scrambling space. Our goal focuses on searching for the best \( h_1 \) as the multiplier in a linear congruential generator, \( \pi_p(n_j) = h_1 n_j \pmod{b} \), so that we can find the best permutation on the set \( n_j \in \{1, 2, \ldots, b - 1\} \).

There are several theoretical procedures to make this assessment, and the spectral test and discrepancy are commonly used criteria. Assume that \( b \) is small, and thus the spectral test [11] is not suitable. Instead consider using the \( L_2 \)-discrepancy, \( D_N^{(2)} \). For a prime modulus \( b \), and a primitive root \( h_1 \) modulo \( b \) as multiplier, we have that the discrepancy, \( D_N^{(2)} \), of the associated linear congruential generator satisfies [12]

\[
(b - 1)D_{b-1}^{(2)} \leq 2 + \sum_{i=1}^{q} a_i,
\]

(5)

where \( a_i \) is the \( i \)th digit in the continued fraction expansion of \( h_1/b \) with \( a_q = 1 \). So our job is reduced to finding a primitive root \( h_1 \) modulo \( b \) such that \( h_1 \) has the smallest sum of continued fraction expansion digits with \( h_1/b = [a_1, a_2, \ldots, a_q] \) and \( a_q = 1 \). A table for the best primitive root modulo \( b \) based on this criterion is listed in [13]. Then \( g_2 \) is chosen to be a primitive root modulo \( b \) such that \( g_1 \) has the second smallest sum of continued fraction expansion \( q_2/b = [a_1, a_2, \ldots, a_q] \) and \( a_q = 1 \). In Figure 1, the right figure is an optimal Faure sequence with \( h_1 = 28 \) and \( g_2 = 83 \).

In addition, error estimation can be obtained by using several scrambled optimal Faure sequences, and each scrambled optimal Faure sequence can be produced by assigning \( g_i \), for \( 3 \leq i \leq m \), in \( A^{(j)} \) in (4) to numbers randomly chosen from the set \( \{0, 1, 2, \ldots, b - 1\} \).

4 Geometric Asian options and numerical results

In this section, we examine the valuation of a complex option for which there is a simple analytical solution. The popular example for such problems is a European
call option on the geometric mean of several assets, sometimes called a geometric Asian option. Let \( K \) be the strike price at the maturity date, \( T \). Then the geometric mean of \( N \) assets is defined as

\[
G = \left( \prod_{i=1}^{N} S_i \right)^{\frac{1}{N}},
\]

where \( S_i \) is the \( i \)th asset price. Thus the payoff of this call option at maturity can be expressed as

\[
\max(0, G - K).
\]

Boyle [14] proposed an analytical solution for the price of a geometric Asian option. The basic idea is that the product of lognormally distributed variables is also lognormally distributed. This is due to the fact that the behavior of an asset price, \( S_i \), follows geometric Brownian motion [15]. The formula for using the Black-Scholes equation [16, 15] to evaluate a European call option can be represented by:

\[
C_T = S \* \text{Norm}(d_1) - K \* e^{-r(T-t)} \* \text{Norm}(d_2),
\]

with

\[
d_1 = \frac{\ln(S/K) + (r + \sigma^2)(T-t)}{\sigma\sqrt{T-t}},
\]

\[
d_2 = d_1 - \sigma\sqrt{T-t},
\]

where \( t \) is current time, and \( r \) is risk-free rate of interest that is constant in the Black-Scholes world. \( \text{Norm}(x) \) is the cumulative normal distribution. Since there exists an analytical solution for a geometric Asian option, this offers us a benchmark to compare our simulation results. The parameters used for our numerical studies are as follows:

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Number of assets ( N )</td>
<td></td>
</tr>
<tr>
<td>Initial asset prices ( S_i(0) ) for ( i = 1, 2, ..., N )</td>
<td>100</td>
</tr>
<tr>
<td>Volatilities ( \sigma_i )</td>
<td>0.3</td>
</tr>
<tr>
<td>Correlations ( \rho_{ij} ) for ( i &lt; j )</td>
<td>0.5</td>
</tr>
<tr>
<td>Strike price ( K )</td>
<td>100</td>
</tr>
<tr>
<td>Risk-free rate ( r )</td>
<td>10%</td>
</tr>
<tr>
<td>Time to maturity ( T )</td>
<td>1 year</td>
</tr>
</tbody>
</table>

The formula to compute the analytic solution for a geometric Asian option is computed by a modified Black-Scholes formula. Using the Black-Scholes formula, the call price can be computed by equation (7) with the modified parameters, \( S \) and
\[ S = Ge^{(-A/2+\sigma^2/2)T} \]
\[ A = \frac{1}{N} \sum_{i=1}^{N} \sigma_i^2 \]
\[ \sigma^2 = \frac{1}{N^2} \sum_{i=1}^{N} \sum_{j=i}^{N} \rho_{ij} \sigma_i \sigma_j. \]

We follow the above formula and compute the prices for different values of \( N \) and list the results in Table 1.

The numerical results are reported in the rest of this section. For each simulation, we have an analytical solution, so we compute the relative error between that and our simulated solution with the formula

\[ \frac{|p_{qmc} - p|}{p}, \]

where \( p \) is the analytical solution in Table 1 and \( p_{qmc} \) is the price obtained by simulation. For different \( N \), the \( p_{qmc} \) is obtained by simulating the asset price fluctuations using geometric Brownian motion. The results are shown in Figure 2, where the label “Faure” refers to the original Faure sequence [17], while “dFaure” refers to our optimal Faure sequence.

From equation (7), we can see that we have to use random variables sampling from normal distribution. Each Faure point must be transformed into a normal variable. The favored transformation method for quasirandom numbers is the inverse

<table>
<thead>
<tr>
<th>( N )</th>
<th>( K )</th>
<th>Analytic Solution ( p )</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>100</td>
<td>13.771</td>
</tr>
<tr>
<td>50</td>
<td>100</td>
<td>12.223</td>
</tr>
</tbody>
</table>

Figure 2: Geometric Mean Asian Options.
of the cumulative normal distribution function. The inverse normal function provided by Moro [18] is used in our numerical studies.

From Figure 2, it is easily seen that the optimal Faure sequence and the original Faure sequence have the same performance when the number of dimensions is as low as 3. However, when the number of dimensions increases to 50, the optimal Faure sequence has better performances than the original Faure sequence.

5 Conclusions

There are numerous applications for the quasi-Monte Carlo approach in computational finance. Unfortunately, there are only a few types quasirandom sequences widely available. Derandomization provides more choices by which to find suitable quasirandom sequences. In this paper, we focused on finding the optimal Faure sequence within $GFaure$. Based on Tezuka’s $i$-binomial scrambling, we proposed an algorithm and found an optimal Faure sequence within the family. We applied this sequence to evaluate a complex security and found promising results even for high dimensions.

References


