A statistical deterministic implied volatility model

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Abstract

We consider the implied volatility surface to characterise agents belief of the future evolution of stock price returns. However, today's market prices do not provide us with the right future anticipations of the stock price process. This is because the implied volatility surface is neither stationary nor Markovian. It is therefore natural to model the evolution of the implied volatility surface directly.

Our goal is to model the implied volatility surface with general dynamics by relating its future evolution to an observable stochastic process and by adding noises. We choose to link the stock price process to the implied volatility which implies that the volatility surface is dynamically modified according to stock price realisations.

We model the stock price process discretely and using conditional expectations we define its joint distributions. We calibrate the transition matrices to historical data augmenting our filtration set by adding past vanilla option prices. To satisfy the absence of arbitrage opportunities, we will impose that future smile surfaces are compatible with today's prices of calls and puts. Also, defining a deterministic smile surface means defining a future deterministic density for stock process. Therefore, a natural condition would be to impose that the future density is actually a conditional density. That is what we will formalise as the Kolmogorov-compatibility condition.

We then apply our approach to the pricing of forward start and cliquet options and show that the forward volatility has to be higher than the spot volatility because of the risk attached to such products which is not taken into account in today's information.

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1 Forward start call option pricing

In this section, we provide the implied volatility surface with general dynamics. It means that we no longer consider the future implied volatility surface static *i.e.* a function of today's one. We link the future implied volatility to an observable stochastic process. A natural idea is to have the surface depends on the stock price process. Strong of this result we derive the consequences of such an hypothesis on *European forward start call option* and *cliquet* pricing. The problem with pricing such derivatives is actually that today's market prices do not provide us with the future anticipations of the stock price processes. This is because the implied volatility surface is neither stationary nor Markovian. Basically, it means that if we denote by t_0 today's date and t_1 and t_2 two future dates with $t_2 > t_1 > t_0$, market prices do not directly give us the probability density function (*pdf*) $\Phi(S_{t_2}|\mathcal{F}_{t_1})$ where $\mathcal{F}_{t_1}$ can be considered all available information at time t_1. Yet the knowledge of this function is of fundamental importance if we want to price forward start call options.

Rebonato [8] explains that if no arbitrage violation is to be allowed there is an infinity of solutions for the future conditional deterministic density, which means that the future is unknown and cannot be derived from today's information. Therefore, in the rest of the article we will look at a general form of smile dynamics and deduce implications on its pdf form.

We place ourselves in a perfect market where there is a riskfree investment opportunity with constant interest rate r and where an underlying stock and plain-vanilla calls of all maturities and strikes are traded. The strikes span a continuum of values, whereas the maturities of plain-vanilla calls and put belong to a set $\{t_i\}_{i=0,\ldots,N}$ The model is set up in a probability space $(\Omega,\ (\mathcal{F}_t)_{t\geq 0},\ \mathbb{Q})$. $\mathbb{Q}$ is the martingale measure under which discounted price process are martingales.

1.1 Pricing

Let's denote $C^F(t_0, t_1, k, T)$ the value a time t_0 of forward start call starting at time t_1 with risky strike kS_{t_1} and maturity T. The No Arbitrage conditions lead to

$$C^F(t_0, t_1, k, T) = e^{-r(T-t_0)}\mathbb{E}\left((S_T - kS_{t_1})^+ |\, \mathcal{F}_{t_0}\right)$$

if we look at forward start call option with pay-off $(S_T - kS_{t_1})^+$.

By conditioning the forward start option with information available at time t_1 *i.e.* by $\mathcal{F}_{t_1}$ we get

$$C^F(t_0, t_1, k, T) = e^{-r(T-t_0)}\mathbb{E}\left(\mathbb{E}\left((S_T - kS_{t_1})^+ |\, \mathcal{F}_{t_1}\right) |\, \mathcal{F}_{t_0}\right)$$

We now consider a general form of implied volatility surfaces. We assume that the implied volatility is stationary in time once it has been related to the stock price process. It means that the implied volatility surface is only driven by the stock process S and time to maturity $T - t$.

$$(\mathcal{H}) \quad \forall\, T \quad \forall t < T, \quad \forall\, K \quad \Sigma_t(K, T) = \Sigma(S_t; K, T - t). \tag{1.1}$$

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Recall that the implied volatility for a given strike and maturity is the number that plugged in Black and Scholes formula fit the European call market prices.

So, our assumption may be rewritten

$$(\mathcal{H})\forall T\ \forall t < T,\ \forall\ K \quad C(t, S_t, K, T-t) = C_{BS}(S_t, K, T-t, \Sigma(S_t, K, T-t)) \tag{1.2}$$

Note that this hypothesis implies that European call prices are not time-dependent *i.e.* do not depend on current time but are stock dependent.

As there exists a one-to-one relation between the implied volatility surface Σ and the pdf ϕ with respect to the Lebesgue measure λ , defined as the function such that for any set $\mathcal{S}$ the probability that $S_{t_1} \in \mathcal{S}$ at t_1 given S_{t_0} at t_0 is

$$\int_{\mathcal{S}} \phi(S_{t_1}, t_1, S_{t_0}, t_0) dS_{t_1}$$

the current stock evolution directly influences its future increment pdf, which means that economic agent anticipations are *dynamic*. Linking stock prices and smile is to say that the smile effect, which characterizes agent beliefs for future evolution of the stock, is dynamically modified according to stock realisations. This is a way of modelling dynamic, *i.e.* rational anticipations.

A consequence of such an hypothesis (1.1) is that call prices are all determined by stock level and consequently that the stock process S holds the whole market risk. Basically, the σ-algebra $\mathcal{F}_t$, which is *a priori* the σ-algebra engendered by $\{\eta_u, u \leq t\}$ where $\eta_u = \{S_u \cup \bigcup_{K,T} C(u, S_u, K, T-u)\}$ coincides with $\sigma(S_u,\ u \leq t)$ for η_u is fully determined by S_u :$\mathcal{F}_t = \sigma\{\eta_u, u \leq t\}$

Therefore $C^F(t_0, t_1, k, T) = e^{-r(T-t_0)}\mathbb{E}\left(\mathbb{E}\left((S_T - kS_{t_1})^+ |\, S_{t_1}\right) |\, S_{t_0}\right)$. Introducing the pdf ϕ we get

$$C^F(t_0, t_1, k, T) = e^{-r(T-t_0)} \int \mathbb{E}((S_T - kS_{t_1})^+ |\, S_{t_1} = x_1)\phi(x_1, t_1, S_{t_0}, t_0)dx_1 \tag{1.3}$$

Now, $\mathbb{E}\left((S_T - kx_1)^+ |\, S_{t_1} = x_1\right) = \int (x_T - kx_1)^+ \phi(x_T, T, x_1, t_1)dx_T$. As we linked the implied volatility surface Σ to the stock level S_t and the time to maturity $T-t$ and as there exists a one-to-one relation between the implied volatility surface Σ and the pdf ϕ, we can consider ϕ a function $\phi(.,\ T,\ S_t, t)$ of stock level and time to maturity. It actually means that, given S_{t_1}, $\phi(., T, S_{t_1}, t_1)$ is *known*.

We can now compute the quantity $\mathbb{E}\left((S_T - kx_1)^+ |\, S_{t_1} = x_1\right)$ for any moneyness k. Speaking in Black and Scholes words, we have:

$$\mathbb{E}((S_T - kx_1)^+ |\, S_{t_1} = x_1) = e^{r(T-t_1)} C_{BS}(x_1; kx_1, T-t_1; r, \Sigma(x_1; kx_1, T-t_1))$$

Therefore we can rewrite Equation (1.3)

$$\begin{aligned} C^F(t_0, t_1, k, T) &= e^{-r(t_1-t_0)} \int C_{BS}(x_1; kx_1, T-t_1; r, \Sigma(x_1; kx_1, T-t_1)) \\ &\qquad \times \phi(x_1, t_1, S_{t_0}, t_0)dx_1 \\ &= e^{-r(t_1-t_0)}\mathbb{E}[C_{BS}(S_{t_1}; kS_{t_1}, T-t_1; r, \Sigma(S_{t_1}; kS_{t_1}, T-t_1))] \end{aligned}$$

which implies that the forward start call option price is given by the discounted expectancy under pricing measure $\mathbb{Q}$ of future European calls prices starting at t_1 and fixing at T. Note that these products do not exist and cannot be priced today unless we make assumptions on future anticipations. This is actually what we are doing here.

***Remark* 1.1** *In a Black and Scholes world, we assume that the increments of processes* $\ln(S)$ *are stationary and independent. Therefore, classic call and forward start call with same time to maturity and same moneyness have same value.*

1.2 Point on Kolmogorov-compatibility

To avoid arbitrage opportunities, it is natural to impose that future smile surfaces are compatible with today's prices of calls and puts. Since defining a deterministic smile surface $\sigma(t, T, K, S_t)$ means defining a future deterministic density ϕ for stock process, we would impose that the future pdf is actually a conditional density. Without doing any assumptions apart from that of no arbitrage opportunity, using the relation between prices and pdf we have $C(t, K, t_1 - t_0) = e^{-r(t_1-t_0)}\phi(K, t_1, S_{t_0}, t_0)$

By definition $C(t, K, t_1 - t_0) = C_{BS}(S_{t_0}; K, t_1 - t_0; r, \Sigma(S_{t_0}; K, t_1 - t_0))$ we have

$$\frac{\partial^2 C}{\partial K^2} = \frac{\partial^2 C_{BS}}{\partial K^2} + \frac{\partial^2 C_{BS}}{\partial K \partial \Sigma}\frac{\partial \Sigma}{\partial K} + \left[\frac{\partial^2 C_{BS}}{\partial K \partial \Sigma} + \frac{\partial^2 C_{BS}}{\partial K^2}\frac{\partial \Sigma}{\partial K}\right]\frac{\partial \Sigma}{\partial K} + \frac{\partial C_{BS}}{\partial \Sigma}\frac{\partial^2 \Sigma}{\partial K^2}$$

For the sake of clarity we denote by Θ the right-hand-side operator.

We will say that if a future deterministic conditional density (or smile surface) defined as

$$\Theta(S_0; K, T - t_0; r, \sigma(t_0, T, K, S_{t_0})) = \int \Theta(S_0; K, t_1 - t_0; r, \sigma(t_0, t_1, K, S_{t_0})) \times\Theta(S_{t_1}; K, T - t_1; r, \sigma(t_1, T, K, S_{t_1}))dS_{t_1} \tag{1.4}$$

is satisfied, then it is a Kolmogorov-compatible density. Therefore, given a current admissible smile surface, if the future smile surface is Kolmogorov-compatible no model-independent strategy can generate arbitrage profits.

There is in general an infinity of solutions for $\Theta(S_{t_1}; K, T - t_1; r, \sigma)$ such that Equation (1.4) is satisfied. Therefore, even if we require the smile surface to be deterministic, there still exists an infinity of future smile surfaces compatible with today's prices of calls and puts. From the constraint previously defined, the conditional density should be chosen so has to be consistent with its historical level.

2 Statistical dynamics

In this section, we model the implied volatility Σ with dynamics based on a statistical analysis of its behavior through time. We first need to infer a shape from

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the volatility surface in accordance with market observable data. A natural choice would be to do a Taylor expansion up to the second order of the implied volatility surface around the money forward level.

However, we consider the shape defined below for the smile with strikes between 25% and 150%

$$f(t, T-t, X, Y, Z; K) = X_t - Y_t \ln\left(\frac{KB_T(t)}{S_t}\right) + Z_t \left(\ln\left(\frac{KB_T(t)}{S_t}\right)\right)^2 \quad (2.5)$$

where X_t, Y_t and Z_t denote respectively the vol ATM, the skew and the curvature of the smile. Of course, the smile needs to be caped in the lowest and highest strikes in order to avoid any arbitrage opportunity.

We need now to provide these parameters with dynamics. The fundamental idea is to say that these three parameters are led by the spot process S_t, for the spot process holds the whole market risk. We then need to infer a shape for the three functions $X(S_t)$, $Y(S_t)$ and $Z(S_t)$.

Plotting X_t, Y_t and Z_t against S_t leads to consider the following models:

$$\begin{aligned} X_t &= \exp(a_x + b_x S_t + \varepsilon_t^x), \\ Y_t &= \exp(a_y + b_y S_t + \varepsilon_t^y), \\ Z_t &= \exp(a_z + b_z S_t + \varepsilon_t^z). \end{aligned}$$

where ε_t^x, ε_t^y and ε_t^z are error processes.

At this point we would need to estimate the model parameters. The ordinary least squares (OLS) means that we suppose that the error processes are white noises and then that an exogen perturbation of the stock has no consequence on the future option values. All the *generalized Durbin-Watson tests* performed on the data reject the hypothesis for white noise. So we introduce dynamics on the errors in order to capture this effect. We now consider the generic model:

$$\left\{ \begin{aligned} \log(\xi_t) &= a^\xi + b^\xi S_t + \varepsilon_t^\xi, \\ \varepsilon_t^\xi &= \rho_1^\xi \varepsilon_{t-1}^\xi + \cdots + \rho_{p^\xi}^\xi \varepsilon_{t-p^\xi}^\xi + u_t^\xi \\ u_t^\xi &\rightsquigarrow \mathcal{WN}(0, (\sigma^\xi)^2) \end{aligned} \right. \quad (2.6)$$

where the notation $u_t \rightsquigarrow \mathcal{WN}(0, \xi^2)$ indicates that u_t is a white noise.

To estimate the parameters of the model we initially fit an high-order model with many autoregressive lags and then sequentially we remove autoregressive parameters until all remaining autoregressive parameters have significant *t-tests*. To fit the model an *exact maximum likelihood method* is used. This method is based on the hypothesis that the white noises are normally distributed, which is in accordance with the Kolmogorov test for normality. Of course, at time t, ε_{t+h}^X, ε_{t+h}^Y and ε_{t+h}^Z are not observable. So at time t we cannot observe X_{t+h}, Y_{t+h} and Z_{t+h}. Supposed that we observe implied volatility from time 0 to time t and that we want to compute the smile at time $t+h$. As S is an exogenous variable, the model values will be computed as $\log\left(\hat{\xi}_{t+h}\right) \equiv \mathbb{E}\left(\log\left(\xi_{t+h}\right) \mid (S_k)_{0 \le k \le t+h},\ (\xi_k)_{0 \le k \le t}\right)$ whence $\log(\hat{\xi}_{t+h}) = a_x + b_x S_{t+h} + \mathbb{E}(\varepsilon_{t+h}^\xi \mid \mathcal{F}_t)$.

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Table 1: Results of calibration for the Vol ATM with(MLE) and without(OLS) autoregressive error.

	Vol ATM		Skew		Curve	
	OLS	**MLE**	**OLS**	**MLE**	**OLS**	**MLE**
Total R^2	0.9316	0.985	0.9258	0.9738	0.429	0.671

Table 2: Results of calibration with autoregressive error for the period 2002/08/1 - 2002/12/30.

	Vol ATM		Skew		Curve	
	Estimate	Std Error	Estimate	Std Error	Estimate	Std Error
α	-0.2209	0.0306	-8.7075	0.1232	6.99E-08	4.62E-08
β	-0.000368	0.0000118	-0.000336	0.0000461	-4.60E-12	1.81E-11
θ	-0.1733	0.00595	-0.593	0.0321	-5.15E-08	6.53E-09
ρ_1	-0.8879	0.0366	-0.5455	0.0324		

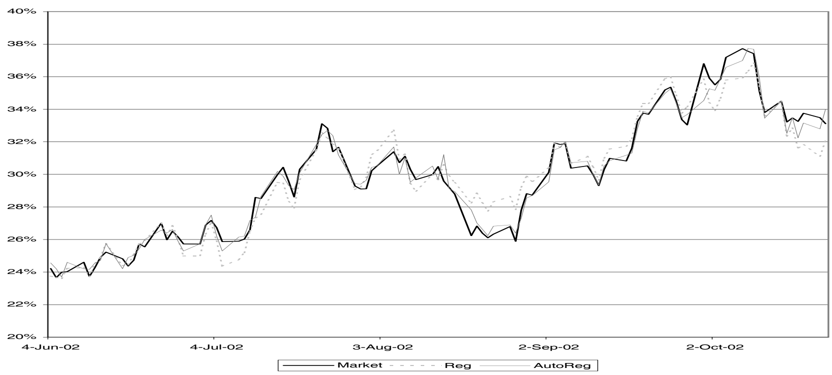

Figure 1: Results of calibration for the Vol ATM with(AutoReg) and without(Reg) autoregressive error for 1 year call options on *Eurostoxx 50* between 2002/06/04 and 2002/12/30.

3 Application to forward start call

In section 1 we derived the calculation of forward start call option prices from the No Arbitrage theory, making the assumption that the smile surface was a deterministic function of the stock price process. In section 2 we suggested a shape for the smile surface using a statistical method. The link between these two parts is actually not obvious since the pdf ϕ we dealt with in section 1 was relative to the pricing or risk-neutral probability measure $\mathbb{Q}$. Basically the implied volatility surface we submitted in previous section characterises the pdf of the spot process under $\mathbb{Q}$ measure but is a function of the spot process under the *historical* probability or *objective* probability that we denote $\mathbb{P}$. We do need to link Σ with the risk-neutral pdf.

We assume that under the measure $\mathbb{P}$ the underlying stock S can be represented as a stochastic volatility log-normal Brownian motion

$$dS_t = \mu S_t dt + \sigma_t S_t d\widehat{W}_t$$

where σ is an $(\mathcal{F}_t)_{t\geq 0}$-adapted stochastic process that we call the *instantaneous volatility* and $\widehat{W}$ is a $\mathbb{P}$-Brownian motion.

Under the probability measure $\mathbb{Q}$, the stock price S has the following diffusion

$$dS_t = rS_t dt + \sigma_t S_t dW_t$$

where W is a $\mathbb{Q}$-Brownian motion.

3.1 Forward start call pricing

Recall that, if λ denotes the Lebesgue measure, ϕ_r is the function defined by: $\mathbb{Q}(S_{t_1} \in \mathcal{S} |\, S_{t_0}) = \int_{\mathcal{S}} \phi_r(S_{t_1}, t_1, S_{t_0}, t_0) dS_{t_1}$ and that

$$C^F(t_0, t_1, k, T) = e^{-r(T-t_0)} \int \mathbb{E}_{\mathbb{Q}}((S_T - kS_{t_1})^+ |\, S_{t_1}{=}x_1) \phi(x_1, t_1, S_{t_0}, t_0) dx_1$$

for any $\mathcal{S}$ in the σ-field of Borel sets. So, if we denote S^* the process such that under measure $\mathbb{P}$, $dS^*_t = rS^*_t dt + \sigma_t S^*_t d\widehat{W}_t$ with $S^*_{t_0} = S_{t_0}$ we have

$$C^F(t_0, t_1, k, T) = e^{-r(T-t_0)} \mathbb{E}_{\mathbb{P}} \left(\mathbb{E}_{\mathbb{P}} \left((S^*_T - kS^*_{t_1})^+ |\, S^*_{t_1} \right) |S^*_{t_0} \right)$$

$$= e^{-r(T-t_0)} \iint (x^*_T - kx^*_1)^+ \phi^*_r(x^*_T, T, x^*_1, t_1) \phi^*_r(x^*_1, t_1, S^*_{t_0}, t_0) dx^*_T dx^*_1$$

Notice that $\phi^*_r = \phi_r$ λ-a.e. and that $S_t = e^{(\mu-r)(t-t_0)} S^*_t$. Denote ϕ_μ the pdf under measure $\mathbb{P}$ of S. We need now to compute the expression of ϕ_μ. We have easily $\phi_\mu(x) = e^{-(\mu-r)(t_1-t_0)} \phi_r(xe^{-(\mu-r)(t_1-t_0)}, t_1, S_{t_0}, t_0)$. Now to price a forward start call option we will compute

$$C^F(t_0, t_1, k, T) = e^{-r(T-t_0)} \iint (x^*_T - ke^{-(\mu-r)(T-t1)} x_1)^+ \times \phi_\Sigma(x^*_T, T, x_1, t_1) \phi_\mu(x_1, t_1, S_{t_0}, t_0) dx_1 dx_T$$

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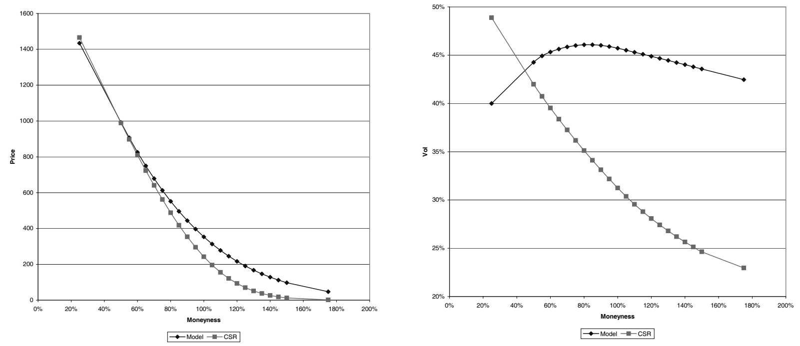

Figure 2: Price and smile for forward start call option on Eurostoxx50 starting in 1Y with maturity 1Y.

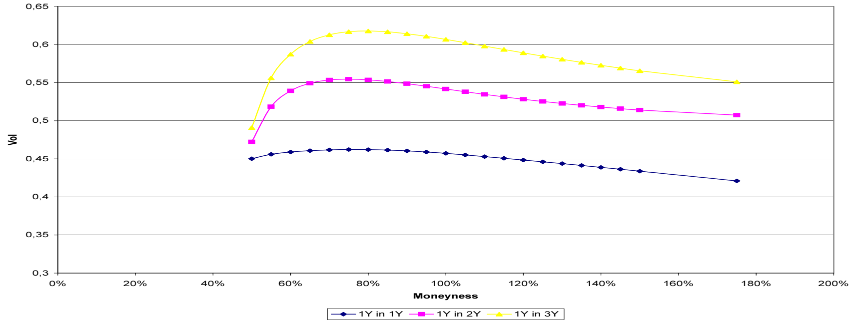

Figure 3: Evolution in time of the 1 Year Forward Implied Volatility surface.

3.2 Calibration

The sole unknown parameter of our model is the *drift* μ of the process S under the historical probability $\mathbb{P}$. To satisfy the Kolmogorov probability, the model must fit European call prices observed at time t_0. This constraint leads to compute the unknown parameter μ. We choose to fit exactly the at-the-money calls. As we have a consistent deterministic implied volatility model which is calibrated to the volatility surface evolution we must fit with a good accuracy the prices for other strike. Of course, we cannot assure that we have an exact fit. To be sure that we satisfy Kolmogorov-Compatibility conditions we have to perturb the estimated parameters while staying in confidence intervals.

3.3 Numerical results

The numerical results (*c.f.* Figure 2) illustrate the fact that the forward smile we obtain using our approach is quite high compared to the spot smile but it is the only forward smile which derived from our model will allow to fit the prices of European call options. Practically, the forward smile does not become flat, moreover the at-the-money volatility tends to increase with time characterizing the increasing risk of owning longer term forward start options, while the skew level remains persistent.

We are now going to explain the reasons why the forward smile is so high. By definition, the Dupire's model is the only model with volatility a deterministic function of time and stock price which presume to know the forward volatility from today's implied volatility surface. It is a model where the volatility surface is assumed Markovian and stationary and which gives a unique solution to the forward volatility. However, empirical tests by Fleming et al (*c.f.* [1]) showed that deterministic volatility models have highly unstable parameters through time. It strongly suggest that this type of models can not explain the time-series variation in option prices. Moreover, the prices of forward start call options given by Dupire's model are lower than the ones observed in the market.

In order to get higher forward smile, practitioners either add a stochastic process to the local volatility or combine the local volatility with a jump process. It means that there is a risk premium attached to a forward start option which is not taken into account in today's information. The problem with Dupire's model is that, while assuming the evolution of the volatility surface to be Markovian and considering an infinity of fixing dates, one cannot control its evolution between the current date and the start date of the forward start option. However, when the number of fixing dates is finite, we have seen that there was an infinity of conditional densities. In our approach, we satisfy this infinity of solution by giving the forward smile a shape consistent with its historical evolution.

4 Conclusion

Our goal was to model implied volatility with dynamics in such a way that vanilla options are priced accurately through time. Observing that market anticipations keep on being adjusted over time according to stock price realisations, we related implied volatility to the stock price process. In a Markov model, the future distributions are conditional on the evolution of the Markov state variable. In order to gain control over such an evolution, we relaxed constraints on the dynamic of the stock price process making it discrete. We chose to calibrate the transition matrices to historical data augmenting our filtration by adding past vanilla option prices.

Consequently, the forward volatility we obtained is much higher than the spot one confirming the fact that market forward start options are underpriced since the risk of a random strike is not taken into account.

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