Different estimators of the underlying asset’s volatility and option pricing errors: parallel Monte-Carlo simulation

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Abstract

This paper investigates accuracy of volatility estimation methods for asset prices characterized by several volatility regimes. Two criteria for evaluation of estimator accuracy are estimation volatility error and option pricing error in the context of the Black-Scholes option-pricing model. We performed a Monte Carlo experiment to obtain average option pricing errors for various patterns of underlying asset volatility. In the case of an ergodic in variance process it is acceptable to use volatility estimates derived either from sample standard deviation of continuously compounded returns or from GARCH-fitted volatility of continuously compounded returns. In the case of a non-ergodic process all estimation methods always yield biased results, in terms of both volatility errors and option pricing errors. Directions of the biases are case-specific.

Keywords: options, pricing errors, volatility, Black-Scholes, Monte-Carlo, parallel computation.

1 Introduction

This paper investigates the impact of using various volatility estimates to price call options. Derivative instruments are traded at prices which are determined by a number of factors, including: current market values of stock prices, T-bill rates, the maturity of the instrument, the exercise price of the instrument, the uncertainty risk (volatility) of the instrument, the liquidity with which the instrument trades, and the premium which is attributable to credit.

The Black-Scholes formula deals with option valuation where equilibrium option price can be found provided that no arbitrage argument holds. In practice,
precise quantification of the Black-Scholes formula is rarely available. The problem lies in the inability to capture the volatility parameter inherent in the stock price data because it is unobservable. There are different ways to measure volatility; here we employed two most popular methods: (i) GARCH-fitted volatility of continuously compounded stock returns and (ii) measuring historical volatility of continuously compounded returns.

In this paper, we looked at costs associated with using both historical volatility of stock price in levels and volatility of stock price in differenced form in call option pricing. The magnitude of the pricing error was obtained by comparing the call option value estimated using the initially generated volatility, further named as actual volatility inherent in the data, with the call option value calculated using historical volatility of stock prices in levels or differences implied in the data.

The remainder of the paper is organized as follows. Section 2 presents an overview of the theory behind the simulation, and underlines the assumptions behind the Black-Scholes model. Section 3 outlines the Monte Carlo technique, experimental design and the parameters from real-life market. Section 4 describes the Monte-Carlo experiments and presents results that we obtained. The final section contains the conclusion.

2 Theory behind the simulation

The Black-Scholes model assumes that the stock price is distributed log-normally with constant variance; therefore, we use geometric Brownian motion (GBM) to generate series for stock prices. GBM is one of the fundamental concepts behind derivative pricing, which essentially describes the behaviour of securities over a period of time. GBM also assumes constant volatility. However, Johnson and Shanno [11] note that there is considerable evidence that variance changes over time. Given these findings, we are going to use several methods of volatility estimation and input results back into the Black-Scholes formula to find out option pricing and volatility pricing errors.

Theoretical formulation of GBM implies that a natural way to estimate the drift and the variance (volatility) parameter of observed stock prices is to run the following regression (proof is available upon request):

\[ \ln S_t = \mu + \rho \ln S_{t-1} + \epsilon_t. \]  

Also, since \( \text{var}\left(\ln S_{t-1}\right) = \text{var}\left(\left(\mu - \frac{1}{2}\sigma^2\right) + \sigma (Z_{t+1})\right) = \sigma^2 \), sample variance of realized continuously compounded stock returns \( r_t = \ln \left( \frac{S_t}{S_{t-1}} \right) \) can be used as an estimate of the corresponding variance rate. Drift and variances rates were estimated for 26 randomly chosen Nasdaq stocks (table with estimation results is available upon request. Historical data on daily prices of these stocks from 1/1/1990 to 12/29/2000 were obtained from Yahoo! Finance, http://finance.yahoo.com/).

Theoretical value of the error can be obtained from the derivative of option price with respect to volatility under the assumption that Black-Scholes pricing model is the correct model; in this study we use the Law of Large Numbers to
assess this error given volatility estimation method and sample size. In particular, we investigate whether the estimation bias varies with the sample size used in volatility estimation or is constant, provided all other option pricing factors are fixed.

The only unobservable volatility parameter in the Black-Scholes formula measures the investor’s uncertainty with respect to the stock returns. It can either be estimated from the history of stock prices (historical volatility as discussed above) or deduced from utilizing the Black-Scholes equation to achieve the implied volatility inherent in the data. Chiras and Manaster [5] found that implied variances were better approximations of future variances than historical variances. Unfortunately, if derivative instruments with respect to certain assets are not available, then computing implied volatility becomes problematic. Such situations may arise in emerging markets or in markets where derivative products are nonexistent or underdeveloped (e.g. real estate). This paper is an attempt to address the issue of volatility estimation for such cases.

3 Monte-Carlo experiments

3.1 Monte Carlo technique

Option pricing via Monte Carlo (MC) can be divided into three basic steps: (1) simulate the stochastic process of the underlying financial assets; each realization is a sample path. (2) evaluate the option value in a backward manner, find the early exercise point and obtain a sample point estimate. (3) average over multiple sample estimates to form an interval estimate that includes a measure of precision. (e.g. standard error). Obviously, the existence of precision measure is one advantage of Monte-Carlo over other numerical methods.

One major difficulty with MC simulation is the large number of simulations required to achieve the required precision measure. Simulations of the order of several thousand is not uncommon in practice. Quasi-MC technique has been proposed [9] to alleviate the simulation error to some extent. Three different ways of parallel implementation of the quasi-MC techniques is reported recently [18] for the pricing problem. We are using a generic MC technique for our simulation.

3.2 Experimental design

In the experiments we are trying to answer the following questions: How much will it cost to assume certain things about the volatility of stock prices? How wrong would results be when we use historical volatility of prices? How wrong would results be when we use GARCH-fitted volatility of continuously compounded returns or historical volatility based on differences of continuously compounded stock returns?

The experiment was initially carried out in statistical package E-Views. In various stages of the experiment volatility of price increments was: (i) Constant; (ii) Increasing; (iii) Decreasing; (iv) Stochastic; (v) Increasing and stochastic;
and (vi) Decreasing and stochastic. The whole idea about using different volatility schemes stems from Schwert [16] and Campbell et al [3], who found that volatilities of individual stock returns increased over the period 1962-1997, while the market volatility did not exhibit any significant pattern. In addition, we developed a MC code in-house and reproduced the results that we got from E-Views. This was used as a launching pad for the parallel implementation.

Increasing and decreasing volatility was generated in the following way:

\[ \sigma_t = \psi \sigma_{t-1} \]  

(3.1)

Obviously, \( \psi < 1 \) meant decreasing volatility, and \( \psi > 1 \) meant increasing volatility. Stochastic volatility was generated according to the normal distribution:

\[ \sigma_t \sim N(\overline{\sigma}, \sigma^2_\sigma) \]  

(3.2)

Increasing/decreasing and stochastic volatilities were generated according to the following distribution:

\[ \sigma_t \sim N(\overline{\sigma}, \sigma^2_\sigma) \]  

(3.3)

the mean value of which, it turn, was either increasing (\( \psi > 1 \)) or decreasing (\( \psi < 1 \)), as follows from formula (3.4):

\[ \overline{\sigma}_t = \psi \overline{\sigma}_{t-1}. \]  

(3.4)

Using the above mentioned volatilities \( \sigma_t \) and a pseudo-random number generator we generated stock price series that were following geometric Brownian motion:

\[ \ln S_t = \gamma + \delta \ln S_{t-1} + \nu_t. \]  

(3.5)

In (3.5) we have \( \gamma > 0 \) to make sure that prices do not fall below zero and increment is normally distributed:

\[ \nu_t \sim N(0, \sigma^2_\nu) \]  

(3.6)

We assumed continuously compounded interest rate 5% and yield curve flat and deterministic, as it is assumed in Black-Scholes model. Expiration date was set 3 months from starting point (time \( t \)), and exercise price was varying from 5 to 105 with step 20 (In our experiments with in-house developed code two more strike prices were considered up to 145). The starting prices \( P_0 \) in all cases were $5.00. Using these parameters, we calculated call prices for no-dividend paying stocks for each point in time \( t \) using all inputs as known. Next, we calculated option prices with all the same inputs but measured volatility. In the first run of the experiment we estimated conditional volatilities and used them in option pricing formula. In the second run of the experiment we took log differences of stock prices and estimated their sample volatilities (Chesney and Scott [4]). After calculating option prices using known data \( C^{TRUE} \) and option prices using observable and measured data \( C^{MEASURED} \), we calculated the option pricing error \( E \) in the following way:

\[ E = C^{MEASURED} - C^{TRUE}. \]  

(3.7)

This error would give us a dollar estimate of the mistake in case of improper use of volatility in the option pricing formula.
4 Execution of the Monte Carlo experiment

For each run of the experiment we executed 20 iterations. The number is certainly small; however, this is done only as an academic exercise and is not a problem to extend to many iterations if enough memory and high speed processing becomes available. For example, when these experiments were conducted on AMD-K6-II processor it simply did not allow us to execute more iterations. With 6 possible exercise prices, 6 patterns of volatility, and 1000 data points one run of the experiment with 20 iterations would take 4 hours 45 minutes. However, this is avoided by running the in-house developed code on modern processors and later by parallelizing the code. Again as an academic exercise we have limited our iterations to 20.

The first set of experiments is described in section 4.3. The second set of experiments were conducted with historical volatility based on differences of continuously compounded prices (for lack of space we do not present the results form the second set of experiments, they are available upon request) using the formula $\delta \ln S_t = \ln S_t - \ln S_{t-1}$. We can summarize results (the set of figures is available upon request) obtained in both of these experiments that as long as our data generating process is ergodic geometric Brownian motion with parameters of the volatility being constant, or stochastic, it is not a serious error to estimate parameters not only of continuously compounded returns utilizing GARCH-fitted volatility estimates but also of continuously compounded stock price differences and use them as estimates of the volatility of underlying stock.

4.1 Volatility

We first generated constant volatility using pseudo random number generator. The initial volatility was generated from a standard normal distribution, such that

$$\sigma_0^{\text{CONST}} \sim N(0,1) \quad (4.1)$$

Each subsequent data point for constant volatility was equal to the previous:

$$\sigma_t^{\text{CONST}} = \sigma_{t-1}^{\text{CONST}} \quad (4.2)$$

Increasing volatility was generated according to the following equation:

$$\sigma_t = 1.001 \cdot \sigma_{t-1} \quad (4.3)$$

In other words, volatility was increasing by 0.1% per day. A sharper rate of increase would lead to an extremely large number at the end of 1000-day sample, and a smaller rate of increase in volatility would not reveal statistical properties of the price series characteristic to this volatility pattern. Decreasing volatility was generated according to the following equation:

$$\sigma_t = 0.999 \cdot \sigma_{t-1} \quad \ldots \quad (4.4)$$

In other words, volatility was declining by 0.1% per day following the above arguments. Stochastic volatility was generated according to the normal distribution with the following parameters:

$$\sigma_t \sim N(\sigma_t^{\text{CONST}},0.016486^2) \quad (4.5)$$
Taking mean volatility to be the same as generated constant volatility allowed us to compare pricing errors between cases of constant and stochastic volatility when expected stochastic volatility equals to the deterministic constant volatility.

Increasing and decreasing stochastic volatilities were generated according to the following equations, respectively:

\[
\begin{align*}
\sigma_t & \sim N(\bar{\sigma}, 0.016486^2) \\
\bar{\sigma} & = 1.001 \cdot \bar{\sigma}_{t-1}
\end{align*}
\]

(4.6)

and

\[
\begin{align*}
\sigma_t & \sim N(\bar{\sigma}, 0.016486^2) \\
\bar{\sigma} & = 0.999 \cdot \bar{\sigma}_{t-1}
\end{align*}
\]

(4.7)

### 4.2 Price trajectories

Once the volatility series were generated, they were used to generate stock price series. We decided to simplify the problem by dealing with no-dividend stocks only. Therefore, our price trajectories were generated according to the following equation:

\[
\ln S_t = 0.006 + 0.998 \times \ln S_{t-1} + v_t
\]

(4.8)

The value of the drift was taken from average drift displayed by real data. The disturbance term \( v_t \) was generated by distribution: \( v_t \sim N(0, \sigma_t^2) \), where the historical volatility of the disturbance \( \sigma_t \) for the true option prices is the volatility estimates obtained from the GARCH stock price generation equation or from the stock price differences.

We generated six variants of price trajectories reflecting possible shapes of the underlying real volatility series: those displaying constant volatility, increasing volatility, decreasing volatility, stochastic volatility with constant mean, stochastic volatility with increasing mean, and stochastic volatility with decreasing mean value. Historical volatility estimated by price differences or volatility estimated by GARCH is exactly the volatility that characterizes evolution of stock prices. Since we are calibrating the original Black and Scholes linear partial differential equation (derivation of the Black and Scholes equation only assumes that the values of option and stock are risk neutral expected values) and the closed-form solution to it doesn't depend on the risk-neutrality of volatility estimates, we don’t have to find risk-neutral counterparts to our volatility estimates. Thus, volatility is not risk-neutral in our data generating process and it doesn’t have to be [1, 13].

### 4.3 Experimental results with GARCH-fitted volatility of continuously compounded stock returns

After generating stock prices we estimated conditional volatility for each sample using GARCH (1,1) model. These volatilities were used to calculate call prices \( C_{\text{MEASURED}} \). After this we calculated call prices \( C_{\text{TRUE}} \) using actual volatilities used in generating price series. Errors of option pricing were measured according to equation (3.7). During the experiment we were varying exercise price of the
call option. Exercise price X was assigned the following values: $5.00, $25.00, $45.00, $65.00, $85.00, and $105.00. Besides, the drift component of the stock price was $0.006t, where t stands for the number of days. Therefore, we were able to plot call pricing error against various exercise prices and unconditional expectations of stock prices ($E[S_t]=0.006*t$).

In this part of the experiment, GARCH (1,1) model with mean equation \( \ln S_t = \mu + \rho \times \ln S_{t-1} + v_t \) is estimated for sample size \{1,k\}. We generate fitted volatility \( h_t \) and record the last value \( h_k \). This is our input into the Black-Scholes formula for calculating call price \( C^{\text{MEASURED}}_t \). Next, we add one more data point to the stock price series, estimate the GARCH model for sample \{1, k+1\}, and use \( h_{k+1} \) to calculate \( C^{\text{MEASURED}}_{k+1} \). By using this process we generate a series of “measured” call prices \( C^{\text{MEASURED}}_t \), and the other series. We plotted pricing errors on Figure 1 (a)-(f). The graphs show us what sort of data generating process creates larger errors if we use GARCH-fitted volatility of continuously compounded returns. As we can infer from Figure 1, in some cases (for example, cases b and e) option pricing errors grow with the sample size (tables with statistical analysis of estimation and pricing biases are available upon request). This can be attributed to the nonstationarity of option prices: as sample size increases, sample volatility of data approaches infinity. Therefore, we get upward-biased estimates of volatility of stock prices.

![Figure 1](image-url)

(a) Constant volatility  (b) Decreasing volatility  (c) Increasing

(d) Stochastic Volatility  (e) Decreasing and stochastic  (f) Increasing and stochastic

Figure 1: Option pricing errors for stock prices generated using various patterns of volatility, average across series. GARCH-fitted volatility estimates based on continuously compounded returns are used in the generation of option prices.
Comparative statistics of Black-Scholes model shows that call price increases when stock price volatility increases. Therefore, upward-biased estimates of stock price yield upward-biased estimates of the option price. This situation could result in a false belief that there exists a Put-Call-Parity arbitrage strategy based on erroneously calculated call prices.

In the case of constant and stochastic volatility of prices (Fig. 1(a and d)) the situation is not as clear. Option pricing errors seem to be fluctuating around zero on the average. Volatility errors are also calculated (but not presented here) and observed that the pattern that is being followed by the volatility errors is closely related to the pattern observed for the option pricing errors. This means that the choice of volatility parameters in applying Black and Scholes formula is extremely significant in affecting the resulting option prices.

### 4.4 Parallel Implementation of Monte Carlo

This Monte-Carlo experiment was carried out by estimating volatility from the sample standard deviation of returns as well as from the standard deviation obtained using GARCH stock price generation model. The time required by the computer to run the program when the volatility is estimated from the estimates of GARCH standard deviation has increased tremendously, accounting almost for the difference in 20 hours (see Table 1).

<table>
<thead>
<tr>
<th>Part</th>
<th>Actual time, sec</th>
<th>Exercise Prices</th>
<th>Iterations</th>
<th>Time per iteration, sec</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>3044.107</td>
<td>5, 25, 45, 65, 85, 105</td>
<td>20</td>
<td>25.368</td>
</tr>
<tr>
<td>2</td>
<td>79105.228</td>
<td>5, 25, 45, 65, 85, 105</td>
<td>10</td>
<td>1318.420</td>
</tr>
</tbody>
</table>

Note: part 1 uses estimates from sample standard deviation of stock returns, part 2 uses estimates from standard deviation obtained using GARCH. Exercise prices are 5, 25, 45, 65, 85, and 105 dollars. We used Pentium 4M processor and EViews 4 for computations and estimation.

We first reproduced the E-Views results with our in-house developed sequential version of the Monte Carlo simulation code before extending to parallel implementation. We used message passing interface (MPI) libraries to communicate among the various processors in a network of workstations. The algorithms were tested up to eight processors. These workstations have Pentium II processors with 350MHz of CPU speed and 512KB cache. The memory in these machines is 256 MB. We discuss briefly the timing results from the parallel MC experiments.

When we have two processors each processor was assigned the execution of MC simulation corresponding to one strike price. Hence, MC simulation does not pose any serious challenges in terms of implementation. Only when the workload increases to executing eight different strike prices, we notice slight
increase in execution time. Otherwise, we get linear speed up, for example, with 4 processors each running one simulation corresponding a given strike price (one out of four strike prices), the execution is almost same as the execution time with 2 processors (each executing one simulation corresponding to one of the two strike prices) as presented in figure 2. The averaging process after execution of the MC simulation takes almost negligible amount of time except for the eight processor simulation. The slight increment in the execution time is due to the communication required by seven of the eight processors with the eighth processor. This is generally expected in MC simulation.

![Parallel Monte Carlo Results](image)

**Figure 2:** Parallel Monte Carlo simulation for eight different strike prices.

5 Conclusion

In this paper we showed the dollar price of indelicate treatment of Black-Scholes option pricing model using the example of a call option on a no-dividend paying stock. It is known that data generating processes for underlying stocks have stochastic volatility. It is also known that Black-Scholes model assumes that volatility of the stock price at time t is known. We performed Monte Carlo experiments to obtain average option pricing errors for various patterns of underlying asset volatility.

In the first set of experiments we used GARCH-fitted volatility of stock prices as an estimate of true volatility while in the second set we used volatility of continuously compounded stock returns as an estimate of true volatility. We obtained the following results: In the case of ergodic in variance probability distribution it is acceptable to both estimators of volatility. In the case of a non-ergodic process Black-Scholes model yields biased results.

Therefore, the assumption of ergodicity seems to be a very important for option pricing. Relaxing this assumption yields errors.
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References


