On revision of the option-based approach to modeling mortgage securities

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Abstract

Using risk-neutral martingale methods together with the intensity-based approach, this paper develops a framework that not only generalizes but considerably extends the option-based approach. We formalize this approach and propose a new variant which promises a better performance. In particular, estimated transaction costs should be closer to those observed on the market. Our general model is not tied to a particular numerical procedure as are option-based mortgage models in the literature. As an example we show that classical Stanton's model [9] is in fact a variant of a splitting-up, numerical method of the first order applied to our model.
Keywords: mortgage, option-based approach, prepayment intensity.

1 Introduction

The prepayment option is what makes mortgage related securities complicated assets to price. This right has a dramatic effect on valuation by introducing cash flow uncertainty which depends on the borrower's view of possible future opportunities (e.g., the borrower's expectation of the future behavior of the yield curve) to refinance the loan. Mortgage models in the literature can be classified into empirical (e.g., Deng *et al* [2], Schwartz and Torous [8]) and option-based (e.g., Kau *et al* [6], Stanton [9]) groups.

One peculiarity of mortgage modeling is that the observed borrower prepayment behavior is far from being optimal. Stanton's model [9] was the first of the option-based models which recognized non-optimality "both ways" (the borrower can prepay when it is not profitable as well as he/she can fail to prepay when it is financially profitable to do so). His model is the most advanced in the literature so far in a certain sense. Still, this model is based on a binomial procedure and, therefore, is discrete in time.


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The study of the mortgage modeling in the literature is exclusively empirical. This is a consequence of widespread belief that modeling prepayment is an "inexact science." Even a general mortgage model have not been developed yet.

In this paper, we present a general mortgage model and apply it to the option-based approach to advance our understanding of option-based models. In Section 2, we define a mortgage model in a rigorous way. The framework is suitable for different types of mortgages, including adjustable rate and all kinds of fixed rate mortgages (such as graduated payment, growing equity or the most popular level-payment mortgages). Both the option-based and empirical approaches are special cases of our intensity-based approach. In Section 3, we define an option-based model, analyze the way it is specified in the literature and propose a new variant of the option-based approach. Then, in Section 4, we formulate an option-based model in the markovian set-up. Our model is the first continuous-time model to use option-based specification of sub-optimal prepayment. The other discrete-time option-based mortgage models can be viewed as various numerical approximations of versions of our continuous model, thus making it possible to speak about the "effectiveness" of the methods with respect to their continuous counterparts. Section 5 illustrates this on the example of Stanton's model [9].

In summary, our new formulation of the general mortgage model allows one to see a mortgage from a fresh point of view. It contributes to a deeper understanding of underlying processes and help researchers to find and apply related known results in numerical analysis to mortgage modelling.

2 General model

We consider the following contract. A borrower takes a loan of P_0 dollars at time $t = 0$ and assumes the obligation to pay scheduled coupons at rate $c_t \geq 0$ continuously for duration T of the contract. The loan is secured by the collateral of some specified real estate property, which obliges the borrower to make the payments. For a mortgage originated at time s, interest on the principal P_t is compounded according to some contract rate m_t. The coupon payment rate c_t and mortgage rate m_t can depend on time and the current state of the economy. They can be deterministic functions and even stochastic processes (as in adjustable rate mortgages). The mortgage is assumed to be fully amortized, i.e., $P_T = 0$.

The borrower has the right to settle his/her obligation during an interval specified by contract (commercial mortgages, for example, often have a prepayment lockout period) and prepay the outstanding principal P_t in a lump sum. If the borrower prepays then he is forced to pay concomitant transaction costs F_t. We do not consider a possibility of default in this paper.

We formalize our setup by introducing a completed filtered probability space $(\mathbf{\Omega}, \mathbf{G}, \{\mathcal{G}_t\}_{t\geq 0}, \mathbf{Q})$, where the σ-algebra $\mathcal{G}_t$ represents all observations available to an investor at time t, $\mathbf{\Omega}$ is a set of all possible outcomes and $\mathbf{Q}$ is the probability on $\mathbf{G}$ ($\supseteq \bigcup_{t\geq 0} \mathcal{G}_t$). The prepayment time, for which we use notation τ, is then a positive stopping time on this filtered probability space.



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We introduce information concerning only the timing of the prepayment as the filtration $\mathcal{D}_t = \sigma(\mathbb{I}_{\{\tau \le u\}} | u \le t)$. Now given $\mathcal{D}_t$ and the original filtration $\mathcal{G}_t$, we are interested in decomposing $\mathcal{G}_t$ into $\mathcal{D}_t$ and an additional filtration $\mathcal{F}_t$, i.e., we have $\mathcal{G}_t = \mathcal{D}_t \vee \mathcal{F}_t$. We interested in having $\mathcal{F}_t$ be the "minimal" such filtration. In most applications, $\mathcal{F}_t$ is the natural filtration of a process which defines the interest rate and real estate processes. We accept this complementary filtration $\{\mathcal{F}_t\}_{t\ge 0}$ as given. In the financial interpretation, $\{\mathcal{F}_t\}_{t\ge 0}$ is assumed to model the flow of observations available to the lender *prior* to the prepayment time τ. Throughout this paper we assume that all given processes are positive and $\mathcal{F}_t$-progressively measurable. Given this information $\{\mathcal{F}_t\}_{t\ge 0}$, the lender cannot anticipate the prepayment since he/she does not have complete information about the borrower (such as intention to move, to divorce, etc.) and the borrower does not prepay as soon as it is profitable to do so (e.g., Hayre and Rajan [4]). Thus, we assume that prepayment time τ is not an $\mathcal{F}_t$-stopping time. Therefore, the following object is well defined because $Q(\tau > t|\mathcal{F}_t) > 0$ for all t.

Definition 1 *The process* $\Gamma_t = -\ln(1-Q(\tau \le t|\mathcal{F}_t))$ *is called the hazard process of the random time* τ. *Equivalently* $Q(\tau > t|\mathcal{F}_t) = e^{-\Gamma_t}$.

We assume that Γ_t is an increasing process. Because there is no particular convenient or special day for prepayment, Γ_t is assumed to be continuous. Indeed, we make the slightly more restrictive assumption that the process Γ_t is an absolutely continuous process, i.e., $\Gamma_t = \int_0^t \gamma_\theta d\theta$ for some process γ_t, called the intensity of the random time τ. Etymology of the term "intensity" (or "hazard rate") for $\gamma(t)$ came from the fact that for small $\triangle t$ we have $Q(t < \tau \le t + \triangle t|\mathcal{F}_t \vee \{\tau > t\}) = \gamma(t)\triangle t + o(\triangle t)$.

A process r_t will represent a short-term interest rate, so that at any time t it is possible to invest one unit in a deposit account and "roll-over" the proceeds until a later time s for a market value at that time of $e^{\int_t^s r_\theta d\theta}$.

If we assume that no arbitrage opportunity exists on the market, then the time-t mortgage price M_t equals the expected discounted value of the future cash flow, with expectation with respect to some martingale measure $\mathbf{Q}$ (a mortgage can be viewed as a bond which pays P_t upon "default" and this result can be found in the literature on defaultable securities, e.g., Bielecki and Rutkowski [1]). In other words, we have

$$M_t = \mathbf{E}\left[\int_t^{\tau\wedge T} c_s e^{-\int_t^s r_\theta d\theta} ds + \mathbb{I}_{\{t<\tau\le T\}} P_\tau e^{-\int_t^\tau r_\theta d\theta} \,\middle|\, \mathcal{G}_t\right]. \qquad (1)$$

All expectations in the paper are taken under this martingale measure $\mathbf{Q}$ without reminder. The connection between real-world and martingale probability measures will be briefly discussed at the end of this section.

The processes c_t is assumed to be uniformly integrable. Then it easy to show that expectation (1) is well defined and finite.



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The following result is borrowed from the credit-risk literature (e.g., we can reformulate Proposition 8.2.1 from Bielecki and Rutkowski [1]). It removes explicit involvement of τ by employing its intensity process γ_t.

Theorem 1 *The value of the mortgage M_t admits the following representation*

$$M_t = \mathbb{I}_{\{\tau>t\}}\mathbf{E}\left[\int_t^T \big(c_s + P_s\gamma_s\big)e^{-\int_t^s (r_\theta+\gamma_\theta)d\theta}ds \;\middle|\; \mathcal{F}_t\right]. \tag{2}$$

An important role for the option-based approaches is played with the borrower's liability. It is defined the same way as the mortgage price (1) with the only difference that instead of P_t we must have $P_t + F_t$, i.e., the borrower pays the outstanding principal plus the transaction costs F_t upon prepayment. F_t is assumed to be uniformly integrable.

Note on discrete coupon payments.
If one assumes discrete coupon payments, then the analogous result can follow from the same Proposition 8.2.1 in Bielecki and Rutkowski [1]) with a dividend process specified as a step-function. This would be the framework, for example, with (non-optimal prepayment extension of) the set-up of Kau, Keenan, Muller and Epperson [6]. It can be analyzed along the same lines in what follows.

Note on the real world and martingale measures.
In the literature on mortgage valuation, authors often capture the stochastic nature of prepayments with an empirically estimated prepayment function of some state process such as the borrower's prepayment incentive, loan-to-value ratio, interest rates, etc. The common feature of this literature is that the prepayment function is assumed to be the same under the real world and martingale measures. Jarrow, Lando and Yu, in their work [5] on default risk, argue that this equivalence is an example of an implicitly applied assumption of conditional diversification. Briefly (and rephrasing the authors so our wording is in terms of mortgage prepayment rather then default), the notion of conditional diversifiability requires that conditioning on the evolution of the state processes, the prepayment processes of borrowers are independent of each other. This captures the idea that once the systematic parts of prepayment risk have been isolated, the residual parts represent idiosyncratic, or borrower-specific, shocks that are uncorrelated across borrowers. Examples of such shocks include divorce, acquisition of a new job or its loss, advent of a new family member, etc. As the nonsystematic risk is not priced, it justifies the practice of using an empirically estimated prepayment function for valuation purposes.

We add that the creation of mortgage-backed securities, namely, the practice of pooling individual mortgages, can be seen as the "physical" implementation of the above argument. As such, we view the prepayment intensity γ_t as a model for prepayment. In general, we state "mortgage model" = "intensity model" = "prepayment model."



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3 Specification of refinancing incentive

As we can see from (2), γ_t determines the price of a mortgage. Let Π_t be a scalar quantity which measures the refinancing incentive of the borrower at time t. This is the most important factor which determines the prepayment rate. To underline it, we explicitly state this dependence, i.e., the prepayment intensity $\gamma_t = \gamma_t(\Pi_t)$. The form of γ_t dependence on Π_t measures how "fast" the borrower is expected to use a refinance opportunity. It is a personal (borrower's) characteristic and is a (important!) matter of statistical research. Once we find $\gamma_t(\cdot)$, it should be stable with respect to changes in economic variables and structural changes in market. These changes are reflected in the refinancing incentive Π_t. Therefore, we understand "mortgage modeling approach" as the way the refinancing incentive is defined.

We define an option-based model as a model where Π_t is a "market price of prepayment profitability." In this section we consider this approach to model the refinancing incentive. An alternative approach can be called the mortgage-rate-based approach, where Π_t is a "comparison" of the contract and current mortgage rates. It is often called "empirical" (e.g., Schwartz and Torous [8]) or, sometimes, "option-based" (e.g., Deng *et al* [2]) too. This approach use a different set of mathematical tools then the option-based approach we defined above (see Goncharov [3] for details).

If a borrower prepays the mortgage, he/she must pay the outstanding principal P_t plus transaction costs and, in return, he/she is liberated from his/her obligations to pay coupons, i.e., he/she "gets back" his/her liability. This idea stands behind the traditional (i.e., used in the literature, e.g., Kau *et al* [6], Stanton [9]) option-based definition of the refinancing incentive $\Pi_t = L_t - P_t$. This specification is flawed because when the borrower refinances he/she repays the outstanding principal with a new mortgage. Therefore he/she assumes a new liability (whose evaluation is based on the new refinancing mortgage rate).

Since the borrower's liability is defined the same way as the mortgage price with $P_t + F_t$ instead of P_t, it can be written as follows

$$L_t^T(m^0) = M_t + \mathbf{E}\left[\int_t^T F_s \gamma_s e^{-\int_t^s (r_\theta + \gamma_\theta) d\theta} ds \;\middle|\; \mathcal{F}_t\right],$$

where m^0 is the mortgage rate contracted at the origination (time $t = 0$) of the mortgage. We skip m^0-dependence on the right side of the equality for notational simplicity. If a mortgage is originated at time t with the initial principal P_t and the new maturity S, then, analogously, we have

$$L_t^S(m^t) = P_t + \mathbf{E}\left[\int_t^S F_s \gamma_s e^{-\int_t^s (r_\theta + \gamma_\theta) d\theta} ds \;\middle|\; \mathcal{F}_t\right].$$

In both representations the second terms represent the price of possible future transaction costs associated with prepayment.



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Using these liability representations we can view the traditional option-based approach as an approximation of the "complete" prepayment incentive $\Pi_t = L_t^T(m^0) - L_t^S(m^t)$. The traditional $\Pi_t = L_t^T(m^0) - P_t$ discards the possibility of paying re-refinancing transaction costs in future and, therefore, overestimates the refinancing incentive. This is one of the reasons why the transaction costs implied by the traditional option-based approach in the literature must be *very* high to explain the prices of mortgage-backed securities on the market (e.g., Stanton [9]). On positive side, the traditional option-based approach removes the necessity to model the mortgage rate process; it requires merely knowledge of the contract mortgage rate.

Having said this it appears to be reasonable to discard the prices of potential transaction costs from both terms. As the result, we get a new refinancing incentive $\Pi_t = M_t - P_t$. This incentive underestimates the transaction costs (since the price of possible re-refinancing transaction costs should fall after refinancing), but has potential to be more "precise" than the traditional approach. Another advantage is that we do not need to evaluate the borrower's liability. Thus, it is simpler numerically and the mortgage price (or value of the mortgage pool) can be utilized to fit the model.

4 Diffusion state formulation

To formulate the mortgage model in a continuous-time diffusion state process setting, which is popular in financial applications, we assume that r_t, c_t, and γ_t are deterministic functions of time t and a state process $X_t = (X_t^1, ..., X_t^n)$, for some n. Additionally, we assume the reduced option-based specification of Π_t, i.e., γ_t also depends on the mortgage price M_t. We comment on the way this framework is applied to the traditional option-based approach at the end of this section. To avoid long expressions we will use notation f_t for $f(t, X_t)$. So that instead of writing $r(t, X_t)$ and $\gamma(t, X_t, M(t, X_t))$ we use notations r_t and $\gamma_t(M_t)$.

The state process X_t is a diffusion process following the stochastic differential equation

$$dX_t = \mu_t dt + \sigma_t dW_t, \quad X_s = x \in D \tag{3}$$

with an m-dimensional Brownian motion $W_t \in \mathbb{R}^m$ and functions $\mu : [0,T] \times D \to \mathbb{R}^n$, $\sigma : [0,T] \times D \to \mathbb{R}^n \times \mathbb{R}^m$, where D is a domain (i.e., open and connected set) in $\mathbb{R}^n$. Let operator $\mathcal{A}$ be the generator of this diffusion state process X_t, i.e.,

$$\mathcal{A} := \frac{1}{2}\sum_{i,j}(\sigma\sigma^T)_{i,j}\frac{\partial^2}{\partial x_i \partial x_j} + \sum_i \mu_i \frac{\partial}{\partial x_i}.$$

Then the Feynman-Kac representation states that to evaluate the mortgage price M_t, i.e., to solve eqn (2), which in the present setting is written as

$$M_t(x) = \mathbf{E}\left[\int_t^T \Big(c_s + P_s\gamma_s(M_s)\Big)e^{-\int_t^s [r_\theta + \gamma_\theta(M_\theta)]d\theta} ds \;\Big|\; X_t = x\right], \tag{4}$$

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one can solve the following backward reaction-diffusion PDE

$$\frac{\partial M}{\partial t} + \mathcal{A}M - rM + c + (P - M)\gamma(M) = 0, \quad x \in D \tag{5}$$

$$M_T(x) = 0, \quad x \in D,$$

where we left out the dependence of M, r, c, γ, P on time t and the value of the state variables x for notational simplicity and to underline the nonlinear dependence of γ on M. Under quite general assumptions applicable to many popular set-ups stated in Goncharov [3] a classical solution $M(t,x)$ (i.e., $M(t,x) \in C^{1,2}([0,T] \times D)$) to eqn (5) exists and is unique.

In the case of the traditional option-based specification (i.e., $\gamma_t = \gamma_t(L_t)$), first we have to solve a liability variant of eqn (5), i.e.,

$$\frac{\partial L}{\partial t} + \mathcal{A}L - rL + c + (P + F - L)\gamma(L) = 0, \tag{6}$$

to find the borrower's liability L. Then, we solve a mortgage price variant of (5) (i.e., with $\gamma(L)$ instead of $\gamma(M)$), which is a linear PDE in this case, to find mortgage price M itself.

5 Stanton's model revised

Using the idea of the splitting-up numerical method (e.g., Marchuk [7]), we can apply the following procedure to solve eqn (6) numerically. Let $\triangle t$ be a time step and $L_{(n+1)}$ be an approximation of $L_{(n+1)\triangle t}$. We go backward in time and determine L_n (the approximation of $L_{n\triangle t}$) in two steps.

The first step: we consider the equation

$$\frac{\partial \hat{L}}{\partial t} + \mathcal{A}\hat{L} - r\hat{L} + c = 0 \tag{7}$$

with the terminal condition $\hat{L}_{(n+1)\triangle t} = L_{(n+1)}$. With its help we evaluate the "fractional" step $L_{n+1/2} = \hat{L}_{n\triangle t}$.

The second step: we consider the remaining part of the original eqn (6), which is the ODE

$$\frac{dL}{dt} - (P + F - L)\gamma(L) = 0 \tag{8}$$

with the terminal condition $L_{(n+1)\triangle t} = L_{n+1/2}$. Finally, we take $L_n = L_{n\triangle t}$.

When solving equations (7) and (8), it is natural to consider only some approximations of the equations since even after exact integration, we get just an approximate solution of the original equation. This particular variant of the splitting-up method is of the *first* order in time.

The first step, as we can see, is really the same as the first step in Stanton's procedure [9] (Stanton solved it using the second order Crank-Nicolson scheme). To see how the second step is related we explicitly solve the following approximation



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of (8) over the interval $[n\triangle t, (n+1)\triangle t]$. We freeze the argument of $\gamma(\cdot)$ at time $t = (n+1)\triangle t$, so that it is known over the interval $[n\triangle t, (n+1)\triangle t]$, i.e., γ is a constant $\gamma(L_{(n+1)\triangle t})$. P and F are frozen at $t = n\triangle t$. Thus we get the ODE with constant coefficients

$$\frac{dL}{dt} + \big(P_{n\triangle t} + F_{n\triangle t} - L\big)\gamma(\hat{L}_{n\triangle t}) = 0,$$

were we used that for the frozen argument of $\gamma(L_{(n+1)\triangle t})$ we have $L_{(n+1)\triangle t} = L_{n+1/2} = \hat{L}_{n\triangle t}$. This equation is easy to integrate analytically. Its solution at time $t = n\triangle t$ is

$$L_n = e^{-\gamma(\hat{L}_n)\triangle t}\hat{L}_n + \big(1 - e^{-\gamma(\hat{L}_n)\triangle t}\big)\big(P_{n\triangle t} + F_{n\triangle t}\big). \tag{9}$$

Now it is easy to recognize the second step in Stanton's econometric procedure [9]. That is L_n is an expectation of the binomial "prepay vs. continue" outcomes (i.e., "$P_{n\triangle t} + F_{n\triangle t}$ vs. $\hat{L}_n$") over time period $[n\triangle t, (n+1)\triangle t]$. Therefore, Stanton's procedure is a variant of the splitting (fractional step) method for numerical solutions of semi-linear PDE (6). It is important to note here that by freezing arguments as above, we make (9) to be the first order approximation of the solution to the equation (8).

We make several important observations. Because the prepayment function has a jump, numerical methods, in general, cannot be of the second order approximation unless the discontinuity is specially treated. Additionally, we can get problems with spurious oscillations. However, a numerical solution of the *liability* equation does not have this problem since in this case the coefficient $(P + F - L)\gamma(L)$ in (6) is Lipschitz continuous because $\gamma(L)$ has a jump exactly at $P + F$. This is not true for the mortgage price PDE. Therefore, researchers should be careful with an application of a jump specification of γ to a reduced option-based model because they deal with non-linear PDE (2) in this case. The traditional option-based model has the same problem though it is not so "dangerous" because the mortgage PDE is linear ($\gamma(L)$ depends on a smooth L).

Even with a smooth specification of γ_t (this is a natural assumption, see Goncharov [3]) Stanton's computational method is of the first order in time because of two reasons. First, the splitting-up method used is of the first order. This can be easily corrected with a little increase in computations (see Marchuk [7]). Second, the second step in Stanton's procedure is of the first order approximation of the original ODE (8). Therefore, any variant of a splitting-up method as a whole would of the first order in time. In addition, instead of working with (9) we can work with some approximation of equation (8) of an appropriate (second) order with the benefit of faster calculations since algebraic operations are performed faster on a computer than evaluations of the exponential functions in (9) (recall that the splitting-up method is an approximation itself).

In summary, knowing the real PDE standing behind an option-based model, we can be more efficient by using the rich literature on numerical analysis. For example, for Stanton's mortgage model we can employ a two-step splitting method (e.g.,



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Marchuk [7]) to get the second order in time at little cost of additional computations.

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