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Nonlinear Reduced Models for Rapid Assessment of Effects

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Abstract

This study aims at integrating results from large-scale models into Decision Support Systems (DSSs), so that the effects of measures can be rapidly assessed for use in the design, evaluation, construction and maintenance of major civil engineering systems. To fulfil this aim these models must be reduced in size and computation time.

In hydroscience, large-scale numerical models are often of order 100,000. Consequently, model simulations require long computation times. In this paper we apply a systematic, generic reduction tool, which can extract the essential patterns from a large data set of model results. The model reduction technique is used to significantly reduce computation time, while maintaining an acceptable level of accuracy.

In an earlier paper [1], large-scale models were reduced to a general linear auto regressive model in reduced space. In this paper we extend this method to nonlinear models, by incorporating knowledge of the original model and of the governing physical processes. In order to reduce the dimension of the system, eigenvalue decomposition of the covariance matrix of the data set is carried out. Only the most important eigenmodes are taken into account. These eigenmodes represent the most significant dynamic spatial patterns in the output of the complex model.

In this paper some interesting results are discussed.

1 Introduction

Modelling and simulating realistic problems is often inhibited by limitations in computer power. In Decision Support Systems (DSSs) the lack of computational power is even more prohibitive, since DSSs generally run on PC's and are required to give "interactive" feedback. With "interactive" it is meant, that the computer should produce results within a few seconds.

This paper sets out to reduce complex models to such a degree that they yield acceptable results on a PC in a time-scale which can be called interactive. Attempts at reducing models by using black-box models or general linear models have proved to be successful in some cases [1]. The problem of these models is that the range of applicability is rather limited. It is therefore desirable that knowledge of the original physical system is used in the reduction process as much as possible. This implies incorporation of non-linearity into the reduction process. Hence the title of this paper: Non-Linear Reduced Models for Rapid Assessment of Effects.

Please note that the focus of this study is to reproduce the complex models, and not necessarily to reproduce the "real" world. However, the same techniques can be applied to "real" world measurement data.

In this paper first an overview of the reduction process is given in Section 2. Possibilities for the introduction of non-linearity are discussed for each phase of the reduction process. Some examples are then described in Section 3, followed by a discussion conclusions in Section 4.

2 Introducing nonlinearity in the Reduction Process

The reduction process, which was adopted in the present study, will be briefly described in this section. In the following sections we will consider data sets which contain the output of complex models. It is assumed that these data sets are time series at grid points, nodes or other entities. The data set, however, need not necessarily be model output, and may just as well be measurement data.

2.1 Identifying the information need

The reduced model can be determined such that it will, on average, reproduce the entire complex model with equal accuracy. However, very often only a limited part of the output of the model is relevant for the problem under consideration. It would then be wise to design and optimise the reduced model such, that it reproduces the desired information with high accuracy only. Variables which are relevant for the decision making process in the Decision Support System are called *decision variables*. For example: a decision variable in case of a dredging problem could be: the average bed level in the main channel.

The output of the complex model, which is at first glance not directly relevant for the problem under consideration, might have, however, a (sig-

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nificant) influence on the decision variables and must also be considered in order to design an accurate reduced model.

This phase requires knowledge of the physical processes underlying the system or the model, even if the rest of the process is completely black-box. Choosing which data to include and which to exclude is not trivial.

2.2 Filtering of the output

If the information need consists of variation on a particular spatial or temporal scale, then all other scales could be filtered out. This is a valid approach if the relevant scales can be assumed to be independent of other scales. Common filters for these purposes are the Fourier Transform or the Wavelet Transform. The filtered data yields "new" data files which will be considered for further analysis and modelling.

Another type of filtering may consist of decomposing the original data set into several separate data sets according to some criteria. For example: separating into geographic coherent data sets. Considering the dredging model, one may choose to separate shallow and deep area's in the model according to some criterion. It is expected that ensuing analysis will yield different results and different spatial and dynamic patterns for each data set. This may result in a more physically based model. The separate data sets can each be locally approximated with a linear model, after which they are integrated into one model. This integration process can be linear (see [5]) or nonlinear (see [4]).

The filtering phase can also consist of various standard filters such as normalization, filtering out a linear trend and so on. Other important filters include non-linear transformation of the data set. Example: sediment transport depends on velocity to some power (say to the power 4). If a relationship between the two must be identified then a transformation of the velocity data by the higher power transform might enhance the analysis and gives more physically meaningful relationships.

2.3 Identifying patterns in the data sets

The main cause of long computation times of complex models is the (very) large number of grid points. The behaviour at the various grid points, however, is often very similar. Pattern identification techniques seek to identify the common dynamic behaviour which occurs at every grid point. This usually yields a limited number of genuinely different dynamic modes. If these modes can be succesfully modeled (see the next section) then the reduced model has a dimension which is equal to the number of relevant patterns. This process is also referred to as "dynamical reduction", or the "remodelling problem" [3].

The main focus in this phase, is to seek two operators: a *reductor* and a *reconstructor*. Another name frequently used for these operators is *encoder* and *decoder*, respectively. The first operator maps the high-dimensional data $X \in \mathbf{R}^n$ to a low dimensional data $Y \in \mathbf{R}^m$ with $m \ll n$, see Equation (1).

$$\mathcal{G} : X \rightarrow Y \tag{1}$$

The second operator performs the reverse mapping by mapping from $Y \in \mathbf{R}^m$ to $\widetilde{X} \in \mathbf{R}^n$, in which $\widetilde{X}$ should approach X. This is also called the reconstruction mapping.

$$\mathcal{H} : Y \rightarrow \widetilde{X} \tag{2}$$

Sequential application of the two operators yields Equation (3).

$$\mathcal{H} \circ \mathcal{G} : X \rightarrow \widetilde{X} \tag{3}$$

Finding the optimal operators involves minimising $\epsilon = \left\| X - \widetilde{X} \right\| = \|X - \mathcal{H}(\mathcal{G}(X))\|$ for a given number of significant modes m.

The high dimensional space which contains the "real" world data set (X) is also called the *Euclidean space*, while the low dimensional data set (Y) is called *reduced space*.

From a theoretical point of view, it is assumed that the data set X is a high-dimensional representation of a low dimensional set of variables Y (also called *latent variables*). The number of relevant modes is then called the "intrinsic" dimensionality of the system, that is, the "real" dimension of the underlying system. This dimension reduction technique is also referred to as embedding or embedology [3]. The name is derived form the concept that, embedded in the large data set, is a small number of modes which can explain the entire data set. Much research has been done in this area in the fields of Nonlinear Analysis and Chaotic Systems (See for instance [6]).

In practise, it is often difficult to distinguish between a genuine physically based pattern and a statistical artifact.

The most commonly used method to find operators $\mathcal{G}$ and $\mathcal{H}$ is identifying patterns in the data with largest variance. This can be easily done by first computing the covariance matrix of the data set, and determine its eigenvalues and eigenvectors. The eigenvectors are the patterns in the data in which a certain amount of variance is exhibited. The eigenvalues which accompany each eigenvector indicate this amount of variance. Only those patterns which explain most of the variance are taken into account and the rest is discarded. This method is also called Principal Component Analysis or PCA (the most commonly used term) [7], Emperical Orthogonal Functions or EOF (in the meteorological literature) [8] or the Karhunen-Loeve Transform or KLT [3]. (It must be noted that there are some subtle difference between the mentioned methods, which are beyond the scope of this paper.) The operators $\mathcal{G}$ and $\mathcal{H}$ are then linear operators given by Equation (4).

$$\mathcal{G} : X \rightarrow \mathbf{G}X = Y, \quad \mathcal{H} : X \rightarrow \mathbf{H}Y = \widetilde{X} \tag{4}$$

$\mathbf{G}$ and $\mathbf{H}$ are matrices. Note that $\mathbf{G} = \mathbf{H}^T$ in which the T denotes the transpose operator.

Nonlinear versions of this type of transform are also known ([3],[7]), but they are not applied as frequently due to the higher computational demand of these methods. They are also more difficult to perform and not as generic in application.

2.4 Developing the reduced model

If the patterns have been identified and the functions $\mathcal{G}$ and $\mathcal{H}$ have been defined, the data in Euclidean space X can be transformed to reduced space by means of Equation (1). This yields data Y in reduced space. The data set Y consists of m time series $y_i(t_k), i = 1..m$ at times $t_k, k = 1..T$. The next step in the reduction process is to find meaningful ways to model these time series.

Simple linear auto-regressive models have been applied in [1]. The model which was identified in this paper is given in Equation (5).

$$\mathbf{y}(t_{k+1}) = \mathbf{B}\mathbf{y}(t_k) + \mathbf{f}(t_k) \tag{5}$$

(Note that vectors in this paper are in bold lower case, matrices in bold upper case, operators, functions and transforms are in caligraphic type face, and data sets in italic uppercase.) Matrix **B** in Equation (5) contains the information on the dynamics of the model and must be determined in the modelling process.

Comparable linear models have also been applied in [8] and [9]. The purpose of the latter references, however, was not to derive a reduced model, but rather to analyse the output with respect to complex models for the most important modes. With that same goal in mind, Hasselmann [10] suggested in 1988 that, ideally, nonlinear models must be used for the reduced model. In later work, only the linear version has received considerable attention (especially by Von Storch et al. [8] and [9]) and is known as the POP method (Principal Oscillation Patterns). The nonlinear version (also called the Principal Interaction Patterns: PIPs) has, to the knowledge of the authors, not yet been applied due to the complexity of the required analysis. The present study takes one step closer to the suggestion of Hasselmann by identifying nonlinear models to the reduced time series in Y.

In the following, models will be identified which assume linearity in the parameters. This is not as restrictive as it may seem, due to the wide range of nonlinear transformations that can be applied. This does, however, exclude the so called *inherently nonlinear* models from the analysis (see [2]). Many functions, however, can be written in polynomial form through a Taylor series expansion. In this paper we will focus on polynomial nonlinearities.

When nonlinearities are introduced, a distinction must be made between an (assumed) nonlinearity in Euclidean space and in reduced space. Firstly, if it is known that in Euclidean space, a nonlinear term $\mathcal{T}(X)$ plays

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an important role, then this nonlinearity is introduced in the reduced first-order autoregressive model through Equation (6).

$$\mathbf{y}(t_{k+1}) = \mathbf{B}\,\mathcal{G}(\mathcal{T}(\mathcal{H}(\mathbf{y}(t_k)))) + \mathbf{f}(t_k) \tag{6}$$

The above equation means that in order to compute $\mathbf{y}(t_{k+1})$, $\mathbf{y}(t_k)$ must first be transformed to Euclidean space (through operator $\mathcal{H}$), then the nonlinear transformation $\mathcal{T}$ is applied and the result is finally transformed back to reduced space by operator $\mathcal{G}$. This seems like a tedious process, but it can be simplified rather easily in ways which are beyond the scope of this paper. Mapping between different spaces in the nonlinear term introduces more and different nonlinear terms in reduced space. An example will be given in Section 3.1.

Secondly, if it may be assumed that the nonlinear term $\mathcal{T}(Y)$ is relevant in reduced space, then the nonlinear term can be easily incorporated into the model as given in Equation (7).

$$\mathbf{y}(t_{k+1}) = \mathbf{B}\,\mathcal{T}(\mathbf{y}(t_k)) + \mathbf{f}(t_k) \tag{7}$$

Of course, more than one nonlinear transformation can be applied.

3 Examples

Two examples will be described in this section. The first is a simple test case model, and the second is a large scale real life case.

3.1 The nonlinear Nash cascade

The Nash cascade was proposed in 1953 by Nash to model rainfall-runoff relationships in catchment areas. It consists of a cascade of reservoirs. Rainfall falls in the top reservoir and the output of the bottom reservoir represents the discharge/runoff. Although Nash originally used a linear model, a nonlinear version will be applied for the present study. The governing set of n differential equations is given in Equation (8).

$$\frac{dh_i}{dt} = c_i\left(h_{i-1}^4 - h_i^4\right) + f_i, \text{ for } i = 1..n \tag{8}$$

Here $h_i(t)$ is the water level in reservoir number i, f_i is the rainfall in reservoir number i (only non-zero for the top reservoir). Coefficients c_i are a combination of several geometric parameters representing the form of the reservoir. This is a highly nonlinear set of equations. In the present context it is more convenient to use matrix notation of the discretised equations.

$$\mathbf{h}(t_{k+1}) = \mathbf{B}_1\mathbf{h}(t_k) + \mathbf{B}_2\mathbf{h}^4(t_k) + \mathbf{f} \tag{9}$$

Simulation with these equations yields a data set. The covariance patterns were determined and collected in matrix $\mathbf{G}$.

Since the original equations are fully known in this case the nonlinearities of the original model can be transferred to the reduced model directly.

$$\mathbf{Gh}(t_{k+1}) = \mathbf{GB}_1\mathbf{G}^T\mathbf{Gh}(t_k) + \mathbf{GB}_2\mathbf{G}^T\mathbf{Gh}^4(t_k) + \mathbf{f}(t_k) \tag{10}$$

$$\underline{\mathbf{h}}(t_{k+1}) = \underline{\mathbf{B}}_1\underline{\mathbf{h}}(t_k) + \underline{\mathbf{B}}_2\hat{\underline{\mathbf{h}}}^4(t_k) + \underline{\mathbf{f}}(t_k) \tag{11}$$

Underlining indicates variables and matrices in reduced space. Special care must be given to the nonlinear term $\hat{\underline{\mathbf{h}}}^4(t_k)$ which is a nonlinear term mapped to reduced space according to Equation (6), and which introduces new nonlinear terms. This is clear when an element $\hat{\underline{h}}_p^4(t_k)$ of $\hat{\underline{\mathbf{h}}}^4(t_k)$ is expressed in terms of $\underline{\mathbf{h}}$. In Equation 12 the g_{ji} are elements of the operator matrix $\mathbf{G}$.

$$\hat{\underline{h}}_p^4(t_k) = \sum_{i=1}^{n} g_{pi}\left(\sum_{j=1}^{m} g_{ji}\underline{h}_j\right)^4 \tag{12}$$

In Equation 13 this is expressed for $n = 10$ and $m = 2$.

$$\hat{\underline{h}}_p^4(t_k) = \sum_{i=1}^{10} g_{pi}\left(g_{1i}^4h_1^4 + 4g_{1i}^3h_1^3g_{2i}h_2 + 6g_{1i}^2h_1^2g_{2i}^2h_2^2 + 4g_{1i}h_1g_{2i}^3h_2^3 + g_{2i}^4h_2^4\right) \tag{13}$$

In Figure 1 the model was reduced from a system of order $n = 10$ to a system of order $m = 4$. Comparable results were obtained for different forcing functions and/or initial conditions.

3.2 Western Scheldt Dredging Model

The second example, which has been considered concerns a real life case. A complex morphological model was used to evaluate various dredging strategies in the Western Scheldt. Each run with the model required one week of computation time on a workstation. Fifteen runs yielded ten CD-Roms of compressed data. An important decision variable was the bed level evolution on a sill in the navigation channel. The original model has a complex set of equations and it was difficult to retrieve the full operator. It was chosen, therefore to identify a new model, which incorporated similar nonlinearities as the original model. The effect of mapping nonlinearities (as given in Equation (6)) to reduced space was taken into account. In this study only bed level data will be considered. It is assumed that a change in the bed will affect the flow linearly. The velocity of flow has an effect on sediment transport which is highly nonlinear (say to the power four). The effect of sediment transport on the change of the bed level is again approximately linear. The relevant nonlinearity is, therefore, an autoregressive effect of the bed level on its own change to the power four. In effect, the same model structure is used as in Section 3.1. However, in the case discussed in Section 3.1 the coefficients of the system matrices of the reduced

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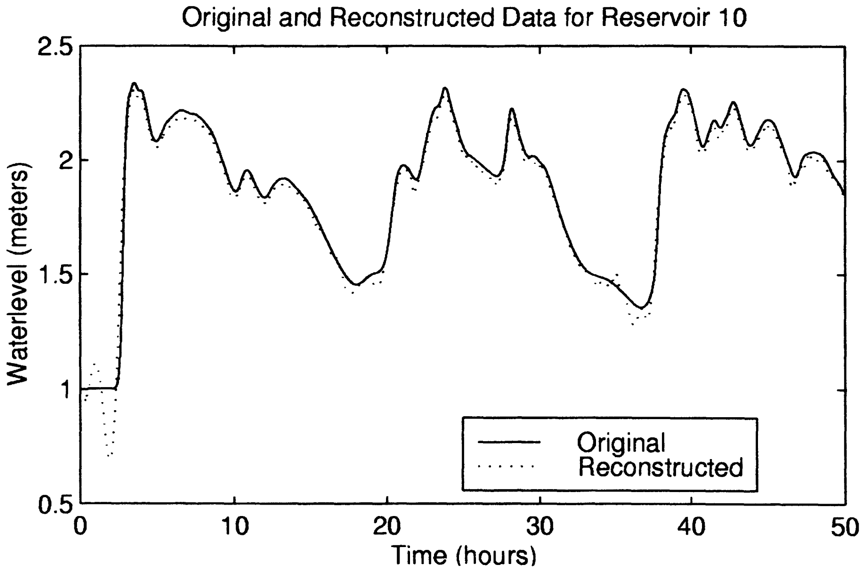

Figure 1: Original model compared to reduced model for reservoir with largest error.

model were determined by direct transformation to reduced space. In the present case the coefficients of the system matrices are determined by an optimisation process.

EOF analysis yielded two significant modes in the data. In Figure 2 the two variables in reduced space (also called Latent Variables (LV)) are plotted.

In Table 1, results are compared for two cases: the first case was used for calibration, and the second case for verification (through prediction). Three models are compared: a linear model as given in Equation (5), a 2^{nd} order model and the above mentioned 4^{th} order model. In Table 1, the difference between the real and the reconstructed bed level evolution on the sill are given.

Case	*Linear Model*	2^{nd} *order Model*	4^{th} *order Model*
Dredge & dump	0.34	0.27	0.20
Dump only	1.5	1.46	1.45

Table 1: Table with root mean square errors for 3 different models for two cases. The first case was used for calibration.

Results indicate a significantly improved fit for the calibration case. The nonlinear models predict the verification case only slightly better than the linear model. The behaviour displayed in the reduced space over a

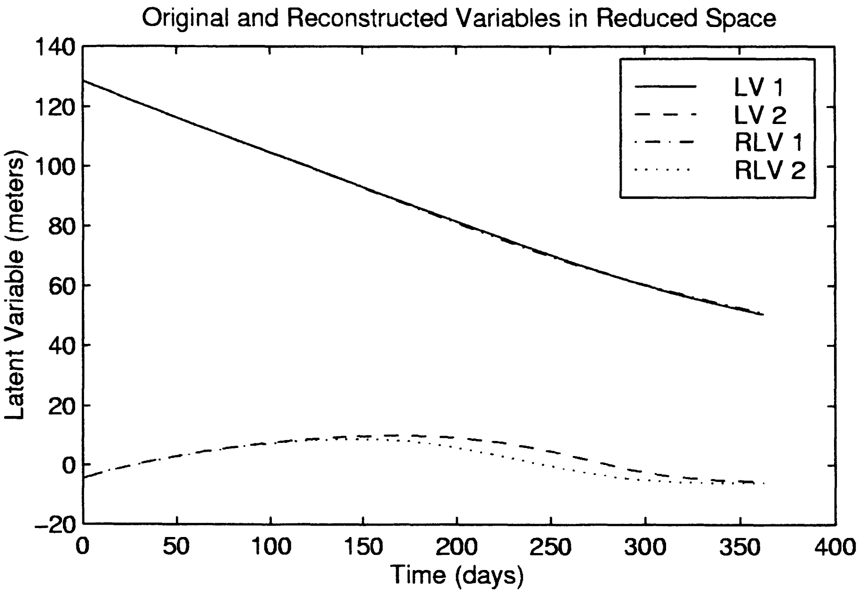

Figure 2: Latent variables for real data and 4th order model. (Note: RLV= Reconstructed Latent Variable)

period of one year is only weakly nonlinear and therefore application of a nonlinear model yields only little effect. In case of complex nonlinear behaviour (such as the Nash cascade of Section 3.1), introducing nonlinearity will improve the accuracy much more.

4 Discussion and Conclusions

This paper shows that nonlinearities can be successfully implemented in reduced models. In the examples of this paper, the nonlinear models performed better than their generic linear counterparts. By including nonlinearities knowledge of the physical system can be incorporated in the reduction process. Although not discussed in this paper, this provides opportunities for evaluating the relative importance of different (nonlinear) terms in a domain, increasing our insight into the complex model. For some cases the reduced model was numerically unstable. The effect of model reduction on numerical stability still requires more research. More work is also needed to study the effect of different methods for identifying patterns (this paper only deals with EOFs).

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