A comparison of high resolution schemes for transcritical shallow water flow

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Abstract

A simple update to a well known classical scheme for solving the shallow water equations is described which is computationally efficient and able to resolve accurately both subcritical and supercritical flows. The solver is the MacCormack scheme with a total variation diminishing term appended to the corrector step which eliminates the numerical oscillations which often arise when the convective terms are discretised using classical central difference schemes. The scheme is explicit, second order in space and time and formulated in finite volume form for ease of implementation on a general boundary conforming grid. Bench mark solutions in one and two dimensions involving both steady and nonsteady flows are shown to illustrate the high spatial accuracy and computational efficiency of the method compared to both classical schemes and modern Riemann-based upwind schemes.

1. Introduction

Numerical models based on the shallow water equations are being used increasingly by the hydraulic engineer as a cost effective way to analyse a wide range of coastal and estuarine flow problems. Discontinuous solutions corresponding to transcritical flow features like bores and hydraulic jumps may be caused by rapid variations in channel topography (e.g estuarine narrowing, sand bars) or by wave diffraction and reflection (e.g current deflecting walls,
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harbour entrances). These give rise to high spatial gradients in depth and velocity, the resolution of which presents a severe challenge to classical numerical schemes. Such schemes exhibit spurious oscillations around flow discontinuities and require artificial viscosity terms to stabilise the solution. In many cases the schemes contain adjustable parameters which require calibration. Modern upwind schemes based on Riemann solvers\(^1\)\(^-\)\(^3\), avoid these problems and resolve high gradients accurately but at extra cost in computational time. The authors present a total variation diminishing (TVD) version of a classical explicit second order accurate scheme (the MacCormack scheme\(^4\)) which removes unwanted non-physical oscillations with little extra computational cost and without any adjustable parameters to calibrate. The scheme is significantly easier to implement than other TVD variants of the MacCormack scheme\(^5\)\(^,\)\(^6\) and is formulated in finite volume form so that complex geometries can be represented accurately with a boundary conforming mesh. In many cases, the results produced compare favourably with modern Riemann-based upwind schemes thus allowing existing codes to be updated economically.

2. Defining Equations

The integral form of the two-dimensional shallow water equations (SWE) taken over a planar region \(A\) with boundary \(S\) and outward pointing unit normal vector \(n\) are,

\[
\frac{\partial}{\partial t} \iiint_A U \, dA + \oint_S H \cdot n \, dS = \iiint_A Q \, dA + \oint_S H_v \cdot n \, dS
\]

(1)

where,

\[
Q = A + B + C + D
\]

and,

\[
A = \begin{pmatrix} 0 \\ (g/\rho)\tau_{yw} \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ (g/\rho)\tau_{yw} \end{pmatrix}, \quad C = \begin{pmatrix} 0 \\ -(g/\rho)\tau_{yf} \end{pmatrix}, \quad D = \begin{pmatrix} 0 \\ \phi g b_x \\ \phi g b_y \end{pmatrix}
\]

\[
U = \begin{pmatrix} \phi u \\ \phi v \end{pmatrix}, \quad H = \begin{pmatrix} \phi q \\ \phi q + 0.5\phi^2 i \end{pmatrix}, \quad H_v = \begin{pmatrix} 0 \\ (\phi/\rho)\sigma_{xx} i + (\phi/\rho)\tau_{xy} j \end{pmatrix}
\]

\[
\phi = g \, h \text{ is the geopotential, } g \text{ is the acceleration due to gravity, } h \text{ is the water depth, } q = u \, i + v \, j \text{ is the water velocity, } f \text{ is the coriolis force, } \rho \text{ is the water density, } \tau_{yw}, \tau_{yw} \text{ are wind shear stresses, } \tau_{yf}, \tau_{yf} \text{ are bed shear stresses, } \sigma_{xx}, \sigma_{yy}, \tau_{xy}, \tau_{yx} \text{ are the normal and shear stresses respectively and } b_x, b_y \text{ are the bed slopes (measured downwards). } U \text{ is the vector of conserved quantities and } H \text{ and } H_v \text{ are the inviscid convective and viscous flux terms respectively.}

Here we concentrate on the convective terms which cause the most difficulty numerically. They are,
\[ \frac{\partial}{\partial t} \int_U dA + \oint \mathbf{H} \cdot \mathbf{n} dS = 0 \quad . \] (2)

3. Finite Volume Method (FVM)

A finite volume form implementation of the numerical scheme is desirable so that the calculations can be carried out on a computational mesh which conforms to the complex boundaries encountered in practice. The physical region over which the equations are to be solved is tessellated by a structured mesh (indexed by \(i, j\)) of quadrilateral cells of area \(A_{ij}\). \(U_{ij}\) is the integral average of \(U\) over cell \(i,j\) located at the cell centre. Since (2) holds for an arbitrary region it can be approximated over each cell as follows,

\[ \frac{\partial U_{i,j}}{\partial t} = -\frac{1}{A_{i,j}} \sum_{k=1}^{4} H_k \cdot S_k \] (3)

where the summation is taken over each side, \(k\), of cell \(i,j\) in turn and \(S_k\) is the outward pointing normal vector to side \(k\) whose magnitude is the length of side \(k\). \(H_k\) are the fluxes defined at the corresponding cell interfaces and based on neighbouring cell centre data.

4. TVD Theory and the TVD MacCormack Scheme

Schemes which are first order accurate in space are formally monotone and do not introduce nonphysical oscillations around bores and hydraulic jumps. However, they are much too dissipative for practical use, smearing the front over many mesh points, so it seems reasonable to use higher order schemes. Godunov\(^7\) showed that schemes of higher than first order cannot be monotone and will always produce spurious oscillations around discontinuities. This led to developments in so-called total variation diminishing (TVD) schemes\(^8\) which 'preserve monotonicity' by limiting oscillations through added nonlinear artificial dissipation terms. The theory was developed for the model one dimensional scalar conservation, or solute transport, equation and later extended to systems of equations like (2).

4.1 TVD Schemes

The one dimensional scalar conservation equation in differential form with wave speed \(v\) is,

\[ \frac{\partial U}{\partial t} + v \frac{\partial U}{\partial x} = 0 \quad . \] (4)
Let $U_i^n$ be the numerical solution to (4) at $x = i \Delta x, \ t = n \Delta t$ where $\Delta x, \Delta t$ are the spatial and time steps respectively. The total variation of the solution is defined by,

$$TV(U_i^n) = \sum_i |U_{i+1}^n - U_i^n|$$

A scheme is TVD if,

$$TV(U_i^{n+1}) \leq TV(U_i^n)$$

Any three point conservative finite difference scheme for solving (4) can be written in the so-called Harten\textsuperscript{8} incremental form,

$$U_i^{n+1} = U_i^n - A_{i-1/2} \Delta U_{i-1/2}^n + B_{i+1/2} \Delta U_{i+1/2}^n$$

where,

$$\Delta U_{i+1/2} = U_{i+1} - U_i, \ \Delta U_{i-1/2} = U_i - U_{i-1}$$

Different schemes result in different forms for the coefficients $A$ and $B$. It can be shown that a necessary and sufficient condition for such a scheme to be TVD is,

$$A_{i-1/2} \geq 0, \ B_{i+1/2} \geq 0, \ 0 \leq A_{i-1/2} + B_{i+1/2} \leq 1$$

Most classical schemes (e.g. the MacCormack scheme) are not TVD and do not satisfy the constraints (8). Sweby\textsuperscript{9} and later Davis\textsuperscript{10} converted a non-TVD, and therefore oscillatory scheme, into a TVD scheme by appending to it terms which modify the coefficients $A$ and $B$ in order to satisfy (8). This involved the idea of a flux limiter which measures the smoothness of the local solution using a ratio of surrounding gradients. The choice of flux limiter is not unique and may result in schemes which differing properties. Following a similar approach, it is possible to construct appropriate TVD correction terms for the MacCormack (MAC) scheme applied to the scalar solute transport equation (4). It then remains to extend the scalar scheme to the coupled SWE (2).

### 4.2 The TVD MacCormack Scheme (MACTVD)

Since (2) are hyperbolic they have real eigenvalues and so can be uncoupled to form a set of scalar equations like (4). In principle, it is then possible to solve (2) by applying a scalar TVD MacCormack scheme to each scalar equation in turn. However, a suitable form of appended terms for a coupled set of equations was derived in finite difference form by Davis\textsuperscript{10} and later in finite volume form by Causon\textsuperscript{11}. Hence we present the finite volume TVD MacCormack scheme for equations (2) in one dimension. Equations (2) may be solved on a 2D structured mesh by applying a sequence of one dimensional schemes via Strang operator splitting\textsuperscript{12}. The MacCormack scheme is a two stage predictor-corrector scheme defined as follows:

**Predictor:**

$$U_i^{n+1} = U_i^n - \frac{\Delta t}{A_i} \left( H_i^n \cdot S_{i+1/2} + H_{i-1}^n \cdot S_{i-1/2} \right)$$
Corrector:  

\[ U_{i}^{n+1} = 0.5 \left( U_{i}^{n} + U_{i+1}^{n+1} - \frac{\Delta t}{A_{i}} \left( H_{i+1}^{n+1} \cdot S_{i+1/2} + H_{i}^{n+1} \cdot S_{i-1/2} \right) \right) \]  

(9b)

The TVD MacCormack scheme is obtained by simply adding a local TVD correction term, \( TVD_{i} \), to the corrector step. The term denoted by \( TVD_{i} \) is defined as follows:

\[ TVD_{i} = \left( G_{i}^{+} + G_{i}^{-} \right) \Delta U_{i+1/2}^{n} - \left( G_{i-1}^{+} + G_{i}^{-} \right) \Delta U_{i-1/2}^{n} \]  

(10a)

where,

\[ \Delta U_{i+1/2}^{n} = U_{i+1}^{n} - U_{i}^{n} \quad \Delta U_{i-1/2}^{n} = U_{i}^{n} - U_{i-1}^{n} \]  

(10b)

\[ G_{i}^{\pm}(r_{i}^{\pm}) = 0.5 \ C(v_{i}) \left[ 1 - f(r_{i}^{\pm}) \right] \]  

(10c)

where \( v_{i} \) is the local Courant number defined as,

\[ v_{i} = A_{i}/\sqrt{\left( q_{i} \cdot S_{i+1/2} + \sqrt{\phi} \ S_{i+1/2} \right)} \]  

(10d)

and

\[ C(x) = \begin{cases} x(1-x) & , x \leq 0 \\ 0.25 & , x > 0 \end{cases} \]  

(10e)

The flux limiter function is,

\[ f(x) = \begin{cases} \min (2x,1) & , x > 0 \\ 0 & , x \leq 0 \end{cases} \]  

(10f)

and the ratio of gradients is defined by,

\[ r_{i}^{+} = \left( \Delta U_{i-1/2}^{n} \cdot \Delta U_{i+1/2}^{n} \right)/\left( \Delta U_{i+1/2}^{n} \cdot \Delta U_{i+1/2}^{n} \right), r_{i}^{-} = \left( \Delta U_{i-1/2}^{n} \cdot \Delta U_{i-1/2}^{n} \right)/\left( \Delta U_{i-1/2}^{n} \cdot \Delta U_{i-1/2}^{n} \right) \]  

(10g)

where the numerator and denominator in each case involve a scalar product over the components of gradients of the solution vector \( U \).

The allowable time step, \( \Delta t \), is defined by the usual CFL condition as,

\[ \Delta t = v \min_{i} \left\{ v_{i} \right\} \]  

(11)

In our calculations \( V \) was taken as 0.9 to ensure stability.

The present TVD MacCormack scheme has two advantages over similar so-called TVD MacCormack finite difference schemes presented recently\(^5,\)\(^6\): the appended TVD term (10a) is derived in a much simpler form and is easier to calculate than the corresponding corrective terms in the other schemes which are cast in terms of right-eigenvectors and Roe's averages, and, because the scheme is formulated in finite volume rather than the alternate finite difference form, it can be implemented on a general boundary-conforming mesh.

5. Numerical Results and Discussion

5.1 Test 1: 1-D Dam Break
This one dimensional problem presents a severe test of any numerical advection scheme because of the discontinuity in the water height. A domain of length 1.0m was discretised using 100 cells. Initially, the water was at rest with height 1.0m in the left half of the domain and 0.5m in the right half. The test was run for 0.05 seconds and transmissive boundary conditions were used at each end of the domain. Figure 1 compares the exact solution for water heights to results obtained by the MacCormack scheme (MAC), MacCormack scheme with TVD correction (MACTVD) and a modern Riemann-based scheme (RBS). The exact solution consists of three regions of constant height separated by a right travelling bore wave and a left travelling depression wave. The MacCormack scheme oscillates wildly between the head of the depression and the start of the bore. As time progresses these oscillations destroy the character of the solution and lead to nonphysical (negative) water heights. In contrast, the MACTVD scheme is non-oscillatory and shows good agreement with the exact solution. In particular the bore wave is resolved over only four cells. The Riemann-based scheme shows only a slight improvement in accuracy over the MACTVD scheme at the expense of a 40% increase in C.P.U. time. Table 1 summarises the performance of the three schemes in terms of accuracy and speed. The following tests are even more stringent problems for which no stable solutions could be obtained with the uncorrected MacCormack scheme; accordingly, it is discounted in the remaining comparisons of solutions.

5.2 Test 2: Steady Supercritical Flow in a Converging Channel

This problem involves steady supercritical flow through a converging channel with its side walls inclined into the direction of flow. Such a flow scenario might arise in practice in a spillway channel. The channel geometry is illustrated in Figure 2. The water is assumed to enter the channel at a constant state with height 1.0m, zero transverse velocity and a Froude number of 2.7. This state was maintained at the left hand (inflow) boundary throughout a time-marched calculation to a steady-state solution. Solid boundary conditions were used at the upper and lower channel walls and transmissive boundary conditions were used at the right hand (outflow) boundary. The channel was symmetrical and defined by the wall deflection angle $\delta$, width $W$ and lengths $L_1$, $L_2$ and $L_3$ (metres). A body fitted mesh was constructed by dividing segments $L_1$, $L_2$ and $L_3$ into 8, 160 and 56 equal intervals respectively (see Fig. 2). This gave a mesh with approximately uniform spacing in the $x$ direction. In the $y$ direction 280 uniformly spaced cells were used. Figure 3 shows the mesh layout. For clarity, the mesh illustrated is much coarser than the one actually used in the computations.

For this bench-mark test problem, a steady state solution can be calculated analytically. At the start of the channel convergence stationary oblique bore waves are formed inclined at $\alpha^0$ to each bank (see point A, Fig. 2). The two bores are reflected at the centre-line of the channel. By a suitable choice of channel geometry and left hand inflow Froude number ($u(gh)^{1/2}$), the reflected
waves can be arranged to meet the opposite bank at the point where the channel ceases its contraction (the necessary parameter settings in this case are given in Fig. 2). In theory, complete wave cancellation can be expected at the end of the channel contraction (see point B, Fig. 2). The net result is a partitioning of the converging channel into three distinct flow regions (1), (2) and (3) each of which is at a known constant state. The MACTVD scheme was run until the maximum relative error between current and previous water heights over the whole mesh was less than a prescribed tolerance, i.e.

\[ \max_{i,j} \left\{ \left( h_{i,j}^{n+1} - h_{i,j}^n \right) / h_{i,j}^n \right\} < tol \]

Normally \( tol \) was set to \( 10^{-5} \). However, it was found that the solution with the MACTVD scheme had essentially converged with the residual oscillating around \( 5 \times 10^{-4} \), so the calculation was terminated after 2000 time steps. This behaviour is a well known property of the MacCormack scheme. In the case of a Riemann-based scheme, the residual can normally be reduced by several orders of magnitude to the level of machine accuracy and the above setting for \( tol \) is appropriate. However, the precision achievable with the MACTVD scheme would be acceptable for most engineering purposes. Local time stepping, rather than a global minimum time step, was used in each case. Figure 4(a) illustrates a shaded contour plot of height which shows that the MACTVD scheme reproduces the main features of the analytical solution. Figure 4(b) shows the results for the Riemann-based scheme, which also required 2000 time steps. Table 1 compares each scheme with the analytical solution (regions are denoted by subscripts). It can be seen that both schemes produce very accurate results but the MACTVD scheme is twice as fast as the Riemann-based scheme, saving four hours of CPU time. In each case almost complete wave cancellation occurs at the end of the contraction. Some evidence of a centred depression remains in both cases but this is an extremely weak feature and is due to the numerical dissipation inherent in each scheme.

5.3 Test 3: Unsteady Bore Diffraction in a Contraction-Expansion Channel

This two dimensional unsteady flow problem presents a severe test of numerical schemes as it includes the complex processes of Mach reflection at the channel walls and multiple bore-on-bore interactions. A channel, symmetric about the \( x \) axis in the \( x-y \) plane, was defined by the equation of its upper bank as follows:

\[
y = \begin{cases} 
1.0 & , -2 \leq x \leq -1 \\
-0.375 \cos(\pi x) + 0.625 & , -1 < x \leq 0 \\
-0.625 \cos(0.5\pi x) + 0.875 & , 0 < x \leq 2 \\
1.5 & , 2 < x \leq 4 .
\end{cases}
\]

A bore wave positioned parallel to the \( y \) axis and initially located at \( x = -1.0 \) propagates from left to right along the channel with bore Froude number \( F_B = 3 \). The initial undisturbed state of the water to the right of the bore was given by \( h_R \)
= 1.0 with zero velocity components, whilst the left hand state was completely specified by $v_L = 0$ and $F_B$ according to,

$$h_L = 0.5 h_R \left( -1 + \sqrt{1 + 8 F_B^2} \right), \quad u_L = F_B \left( 1 - h_R h_L^{-1} \right) \sqrt{g h_R}$$

A body fitted mesh (constructed in a similar way to that in Test 2) was used with 300 cells in the $x$ direction and 200 cells in the $y$ direction. Results at times $t = 0.21$ and 0.45 seconds are shown in Figure 5 as shaded contour plots of height for both the MACTVD ((a), (b)) and corresponding RBS\textsuperscript{1,4} ((c), (d)) schemes. These figures show complex bore on bore interactions and reflections as the incident bore wave passes down the channel. Although this problem does not have an analytical solution the MACTVD scheme captures the expected flow features and resolves the associated high spatial gradients. The MACTVD results compare favourably with a grid independent solution obtained with the RBS scheme\textsuperscript{14} and with other modern high resolution Riemann-based schemes\textsuperscript{2}. Table 1 compares C.P.U. times for the two schemes. For the two runs it was found that the MACTVD scheme was approximately 170\% faster on average than the RBS scheme.

6. Conclusions

It has been shown that the MacCormack scheme may be easily converted into a total variation diminishing scheme by appending appropriate terms to the corrector step. By so doing the new scheme can be used to solve problems involving high spatial gradients such as bore waves and hydraulic jumps which are areas where classical schemes have been found to perform badly (or fail to perform at all). Furthermore, when compared to a modern Riemann-based solver, the MACTVD scheme is considerably more efficient computationally whilst being of sufficient accuracy for most hydraulic engineering problems. Formulation of the MACTVD scheme in finite volume form also permits the use of boundary-conforming meshes for maximum flexibility in practice. The examples shown and description of the TVD correction terms illustrates that in principle it would be possible to convert other existing solvers into TVD schemes and so extend their range of applicability without the considerable expense of rewriting software.

References


3. Zhao, D. H., et al., Approximate Riemann solvers in FVM for 2D


Figure 1. Comparison of numerical solution (squares) with exact solution for a 1-D dam break. a) MAC  b) MACTVD  c) Riemann-based scheme.

Fig 2. Converging Channel showing geometry and regions of constant state.

Figure 3. Schematic computational mesh for the converging channel
Figure 4. Comparison of shaded height contours for steady state supercritical flow in a converging channel. a) MACTVD b) Riemann based scheme.

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<th>Test 1</th>
<th>MAC</th>
<th>MACTVD</th>
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Table 1. Comparison of speed and accuracy of numerical schemes.

Test 1: numerical and exact solutions for the 1-D dam break,
Test 2: numerical and exact solutions for the 2-D converging channel,
Test 3: MACTVD and RBS CPU times for the contraction-expansion channel.

* Computations were performed on a Silicon Graphics 75 Mhz Power challenge L-series machine using version 7.0 of the Fortran 77 compiler.
Figure 5. Shaded contours of height (metres) for bore diffraction in a converging channel at time t (seconds).
MACTVD: a) t = 0.21, b) t = 0.45;  RBS: c) t = 0.21, d) t = 0.45.