On high resolution finite volume modelling of discontinuous solutions of the shallow water equations

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Abstract

An explicit time-marching finite volume method is described for solving the shallow water equations on boundary-fitted grids. The method is robust, flexible, and provides high resolution of flow discontinuities. Such discontinuous flow features correspond physically to phenomena like hydraulic jumps and bore waves caused, for example, by changes in channel geometry or particular tidal and bed conditions. The method is based on an upwind scheme of the Godunov type which are currently popular in aeronautical CFD. It uses cell-centred collocated data which is extrapolated to cell interfaces using a MUSCL reconstruction approach and slope limiters, thereby effectively eliminating the non-physical oscillations which trouble other methods which treat the non-linear advective terms using staggered grid formulations. The resulting Riemann problem, posed at each cell interface, is solved economically using a fast approximate HLL technique. The implementation of the scheme in integral finite volume form allows complex boundary geometries to be represented. The method is applied to number of benchmark steady state and transient test problems in one and two dimensions.

1 Introduction

Increasing attention is now being paid to the coastal environment because of problems like erosion and water pollution which have both economic and social consequences.

In order to understand these processes the hydrological engineer can either construct a physical or a mathematical model of the region of interest. The latter approach is often cheaper and more flexible than the former and is immune to scaling errors. The difficulty with the mathematical approach is
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accurately modelling the underlying physics and solving the resulting set of equations. Often the engineer only requires fairly coarse detail and is satisfied with the prediction of a small number of important parameters like water depth and velocity. However, simplifying assumptions like inviscid flow, may lead to the formation of discontinuous solutions (bore waves) which traditional solvers (Leendertse\textsuperscript{1}) have been unable to deal with. In many cases the flow is required over complex geometries and simple Cartesian meshes do not adequately approximate the boundaries of the physical region leading to another source of error (Falconer\textsuperscript{2}).

In the following, we use the shallow water equations to model flow and present an explicit time-marching algorithm of the Godunov type (Yang\textsuperscript{3}) which accurately resolves discontinuities. The method uses cell-centred collocated data which is extrapolated to cell interfaces using MUSCL reconstruction (Van Leer\textsuperscript{4}) to avoid non-physical under/overshoots. The resulting Riemann problems are solved using a fast approximate technique (Harten\textsuperscript{5}) for computational efficiency. This is turn provides accurate cell interface data for the evaluation of fluxes around each cell area. Second order accuracy in space and time is achieved by using a two-step Runge-Kutta time integrator due to Hancock (Van Leer\textsuperscript{6}, Mingham\textsuperscript{7}).

The implementation of the scheme in finite volume form allows complex geometries to be fitted accurately with a boundary conforming grid without incurring the complexity of an explicit transformation (Pearson\textsuperscript{8}); all that is required is each cell area and an outward pointing ‘side vector’ for each cell face where the latter serve simply to project the flux vector into the local curvilinear coordinate direction.

This implementation of the Riemann solver and the time integrator is more robust and computationally efficient than other recent high resolution solvers based on Roe’s method (Alcrudo\textsuperscript{9}).

2 Shallow Water Equations

In many engineering situations it is sufficient to average out flow information over depth which, on assuming a hydrostatic pressure distribution, leads to the two-dimensional shallow water equations. In integral form these are,

\[ \frac{\partial}{\partial \mathcal{A}} \iint_R \mathbf{U} \, d\mathcal{R} + \iint_R \nabla \cdot \mathbf{H} \, d\mathcal{R} = \iint_R \mathbf{S}(\mathbf{U}) \, d\mathcal{R} \quad (1) \]

\( \mathbf{U} \) is the vector of conserved variables, \( \mathbf{H} = \mathbf{H}(\mathbf{U}) \) is the flux tensor, \( \mathbf{S}(\mathbf{U}) \) is the vector of source terms (friction, bed slope, coriolis force etc.) and \( \mathcal{R} \) is an arbitrary planar region.

Letting \( q \) be the flow velocity then \( q = u \mathbf{i} + v \mathbf{j} \) where \( i, j \) are the Cartesian basis vectors and \( u \) and \( v \) are the depth averaged components of velocity. Defining the geopotential \( \phi = g h \), where \( g \) is the acceleration due to gravity and \( h \) is the height of the water surface above the bed, gives,

\[ \mathbf{U} = (\phi, \phi u, \phi v)^T, \quad \mathbf{H} = \left( \phi q, \phi u q + 0.5 \phi v i, \phi v q + 0.5 \phi v j \right)^T \]
The homogeneous parts of (1) form the flow equations and describes the time evolution of the height and velocity of the water over the physical region of interest in the absence of source terms. These form a system of hyperbolic partial differential equations which admit discontinuous solutions (bore waves) which are difficult to resolve numerically. This paper will concentrate on the flow equations since any numerical scheme for the solution of a more complicated model with source terms must be able to satisfactorily deal with their behaviour. The flow equations are therefore,

$$\frac{\partial}{\partial t} \iint_R U \, dR + \iint_R \nabla \cdot H \, dR = 0 \quad (2)$$

3 The Finite Volume Method

Many coastal and estuarine flows occur over complicated geometries and traditional finite difference approaches based on Cartesian grids give a sawtooth approximation to the boundaries which may lead to unnecessary solution errors. Furthermore, mesh points cannot easily be concentrated in regions of particular interest without enriching the whole mesh and suffering subsequent loss of computational speed. The finite volume method permits the use of arbitrary grids and thus overcomes the previous two problems. Further, all the analysis is performed with respect to the usual Cartesian coordinates. Using the Gauss divergence theorem (2) becomes,

$$\frac{\partial}{\partial t} \iint_R U \, dR + \oint_S H \cdot n \, dS = 0 \quad (3)$$

where $S$ is the boundary of $R$.

The physical region over which the equation is to be solved is tessellated by cells of area $a_{ij}$ indexed by $i, j$. Let $U_{ij}$ be the mean value of $U$ in cell $ij$ (which is located at the centre of the cell). Since (3) holds for an arbitrary region it can be approximated over each cell by,

$$\frac{\partial U_{ij}}{\partial t} = -\frac{1}{a_{ij}} \sum_k H_k \cdot s_k \quad (4)$$

where the sum is taken over each side, $k$, of cell $ij$ and $s_k$ is the outward pointing normal vector to side $k$ whose magnitude is the length of side $k$. $s_k$ are called side vectors. $H_k \cdot s_k$ is the total flux contribution flowing normally through side $k$ of cell $ij$.

4 Numerical Scheme

The numerical algorithm is a two stage second order accurate Godunov type scheme attributed to Hancock (Van Leer[6]). Using the method of operator splitting (Strang[12]), the two dimensional shallow water equations can be solved by combining the solutions to corresponding one dimensional equations. Hence, for clarity the algorithm is given in its one dimensional form (the i
direction) and denoted by $U_{i}^{n+1} = L_{i} (\Delta t) U_{i}^{n}$ where $U_{i}^{n}$ is the value of $U$ in the $i$th cell at time $n\Delta t$ and $\Delta t$ is the time step.

In terms of the computational region, this means restricting attention to a single row or column of cells. Without loss of generality we consider the case of a single row indexed by $i$ with side vectors for cell $i$ denoted by $s_{i-1/2}$ and $s_{i+1/2}$ as shown in Figure 1.

The scheme is,

Predictor: $U_{i}^{n+1/2} = U_{i}^{n} - \frac{\Delta t}{2a_{ij}} (H_{i}^{R} \cdot s_{i+1/2}^{L} + H_{i}^{L} \cdot s_{i+1/2}^{R})$

Corrector: $U_{i}^{n+1} = U_{i}^{n} - \frac{\Delta t}{a_{ij}} (H_{i+1/2}^{*} \cdot s_{i+1/2} + H_{i-1/2}^{*} \cdot s_{i-1/2})$

In the predictor stage $H_{k}$ must be evaluated at the left and right cell interfaces and, since $H$ depends on $U$, it is necessary to estimate $U$ at these points. Let $U_{L}^{L}$, $U_{R}^{R}$ be values at the left and right interfaces of cell $i$ respectively found by extrapolating $U$ from the cell centre using an estimated cell gradient. The corresponding interface fluxes are denoted $H_{L}^{L}$ and $H_{R}^{R}$ respectively. The corrector stage requires $H$ to be estimated in another way as follows. At interface $(i + 1/2)$ there are two values of $H$, namely $U_{R}^{R}$ and $U_{L}^{L}$, respectively.

The allowable time step, $\Delta t$, is defined by $\Delta t = \nu \min \{ \Delta t_{i}, \Delta t_{j} \}$

where,

$\Delta t_{i} = \min_{i,j} \frac{a_{ij}}{q_{ij} \cdot s_{i+1/2,j}^{L} + \sqrt{\phi} \cdot |s_{i+1/2,j}|}$, $\Delta t_{j} = \min_{i,j} \frac{a_{ij}}{q_{ij} \cdot s_{i+1/2,j}^{R} + \sqrt{\phi} \cdot |s_{i+1/2,j}|}$

For stability $\nu$ was taken as 0.95. The scheme was extended to solve the two dimensional problem using the following symmetric operator sequence,

$U_{i,j}^{n+2} = L_{i} (\Delta t) L_{j} (\Delta t) L_{j} (\Delta t) L_{i} (\Delta t) U_{i,j}^{n}$

to maintain second order accuracy in time and space.

### 4.1 Boundary Conditions

Since there can be no flow through a solid boundary $q \cdot s = 0$ and therefore at a solid interface $H = H(\phi)$. The $\phi$ value is taken as that at the neighbouring cell.
Transmissive boundary conditions are obtained by assuming zero gradients for each component of $\bar{U}$ in the neighbouring cell so that $\phi$, $u$, $v$ at a transmissive boundary are the same as at the neighbouring cell centre.

5 Flux Calculations

We now consider how to obtain the interface fluxes for the scheme.

5.1 MUSCL Interpolation

To obtain the interface values $\bar{U}^L_i$ and $\bar{U}^R_i$, $\bar{U}$ is assumed to be piecewise linear in each cell and corresponding gradients are estimated from neighbouring cell values. To avoid non-physical interpolated values (e.g. negative water heights) these gradients must be limited so that estimated interface values do not undershoot or overshoot their adjacent cell centre values. The following steps give one way of doing this and are derived from the classical MUSCL (Monotonic Upstream Schemes for Conservation Laws) reconstruction schemes of Van Leer. For clarity, attention is restricted to the case of Cartesian cells of constant width $\Delta x$.

a) In cell $i$ a gradient $g_i$ for $U_i$ is computed from cells $i-1$ and $i+1$ by linear interpolation.

b) Before linearly extrapolating $U_i$ to left and right cell interfaces each gradient is limited by multiplying by a constant $l_i$.

c) $l_i$ is chosen such that,

Left interface: $\min(U_{i+1},U_i) \leq \bar{U}^L_i = U_i - g_i l_i \Delta x / 2 \leq \max(U_{i+1},U_i)$

Right interface: $\min(U_i,U_{i+1}) \leq \bar{U}^R_i = U_i + g_i l_i \Delta x / 2 \leq \max(U_i,U_{i+1})$

and is chosen to be maximal and $0 < l_i < 1$.

e.g. If $U_{i+1/2}$ overshoots then $l_i = (\max(U_i,U_{i+1}) - \bar{U}_i)/(g_i \Delta x / 2)$

For an undershoot replace max. by min. in the above.

5.2 HLL Approximate Riemann Solver

The Riemann problem can be solved exactly (Toro$^{[10]}$) but is computationally very expensive and, as the numerical scheme requires the solution of a large number of such problems, a more efficient approximate solution is used. The HLL solver developed by Harten$^{[5]}$ is used as it is accurate, robust and computationally efficient. Details are now given and, for clarity, we consider the one dimensional shallow water equations in the $x$ direction and write $F = H \cdot \bar{i}$ whence equation (2) becomes,

$$\int_R \left( \frac{\partial U}{\partial t} + \frac{\partial F}{\partial x} \right) \, dR = 0$$

(5)
Consider the Riemann fan based at the \((i+1/2)\)th interface bounded by the maximum and minimum wave speeds \(e_R\) and \(e_L\) respectively (Figure 2). \(U^*\) can be determined directly from the shallow water equations if these wave speeds are known.

Evaluating equation (5) over the region ABCD with boundary Z and using Green's theorem gives,

\[ \oint F \, dt - U \, dx = 0 \]

which on evaluation gives,

\[
F_R \, \Delta t - U^*(x_R + x_L) - F_L \, \Delta t + U_L \, x_L + U_R \, x_R = 0
\]

but \(x_R = e_R \Delta t\) and \(x_L = -e_L \Delta t\) and hence, after some algebra,

\[
U^* = \frac{F_R - F_L - e_L U_L + e_R U_R}{2} \qquad F^* = \frac{e_R F_L - e_L F_R + e_L (U_R - U_L)}{e_R - e_L}
\]

So simple expressions for \(U^*\) (and hence \(F^*\)) at the \((i+1/2)\)th interface have been derived. It remains to estimate the left and right wave speeds. There are many choices. The 'two expansion' estimates prove to be very robust. These are,

\[
e_L = \min(u_L - \sqrt{\phi_L}, u_s - \sqrt{\phi_s}) \quad e_R = \max(u_R + \sqrt{\phi_R}, u_s + \sqrt{\phi_s})
\]

where,

\[
u_s = \frac{(u_L + u_R)}{2} + \sqrt{\phi_L - \sqrt{\phi_s}} \quad \sqrt{\phi_s} = \left(\sqrt{\phi_L} + \sqrt{\phi_R}\right)/2 - (u_R - u_L)/4
\]

6 Numerical Results

6.1 Validation of the Riemann Solver

Since the approximate Riemann solver is central to our scheme we compare it with the exact Riemann solver in the following one dimensional dam break. An interval of length 1.0m was chosen and discretised using \(M=100\) points so
that $\Delta x = 0.01$ m. Initially the water was at rest i.e. $u(x,0) = 0$ and the left and right heights were such that:

$$\phi(x,t) = 1.0, \quad 0 \leq x \leq 0.5; \quad \phi(x,t) = 0.1, \quad 0.5 < x < 1.0$$

Results after 0.4 seconds are shown in Figure 3 (the dashed line is the numerical solution).

The next two problems use a boundary fitted mesh to accurately represent a channel with a 'bend'. Figure 4 gives the general picture.

### 6.2 Hydraulic Jump

This is a steady state problem in which a flow into a narrowing channel produces a hydraulic jump. In this simulation $A = 8.95^0$, $L = 40$ m, $B = 30$ m, $C = 10$ m and a 40 x 30 grid of cells was used. The left hand inflow boundary state was kept constant with $h = 1$ m, $u = 8.57$, $v = 0$. The results compare favourably with the analytical solution reported in Alcrudo\textsuperscript{[9]} . In particular the angle of the hydraulic jump ($30^0$) is almost exactly reproduced as are the height (1.5 m) and velocity (7.95 m/s) downstream. Figure 5 shows a contour plot of water height.
6.4 Oblique Bore Reflection

This problem presents a severe test for any numerical scheme and is analogous to single Mach reflection in gas dynamics. A bore wave hits an oblique channel wall and undergoes a reflection. Figure 6 is a shaded contour plot of water height after 130 seconds. The scheme clearly resolves the incident bore wave, curved reflected bore front, slip line and bore stem. In this simulation $A = 25^0$, $L = 600$ m, $B = 300$ m, $C = 150$m and a $600 \times 300$ grid of cells was used. The problem is completely specified by the bore Froude number, $F_s$, and the state of the undisturbed region to the right of the bore. We take $F_s = 8$, $\phi_R = 1$, $u_R = v_R = 0$. The left states are calculated from results analogous the Rankine-Hugoniot jump conditions of gas dynamics which state that

$$\phi_L = d \phi_R, \quad v_L = v_R, \quad u_L = S + (u_R - S) / d, \quad d = 0.5 \left( -1 + \sqrt{1 + 8 (\sqrt{F_R} - \sqrt{F_s})^2} \right)$$

where $S$ is the bore speed and $F_R = u_R^2 / \phi_R$, $F_s = S^2 / \phi_R$.

The results closely match those of Toro$^{11}$. 

![Figure 6](image)

6.4 Tidal Bore

This problem simulates tidally induced flow into a narrow estuary such as the Severn. The resulting bore wave forms and sweeps along the estuary and is well resolved by the scheme. Profiles of water height are given at 89, 223 and 357 seconds after low tide in Figure 8. Initially the depth was 5m everywhere and the water is still. The tidal period is 12.4 hours with an amplitude of 3m. The flow geometry is given in Figure 7.
Conclusions

The proposed numerical scheme shows promise for solving the shallow water equations accurately and resolving discontinuous solutions for both time dependent and stationary phenomena (bore waves and hydraulic jumps). The approximate Riemann solver is both accurate and computationally fast. By casting the solver in finite volume form arbitrary boundary geometries can be easily fitted which further increases accuracy.

References


