The wave equation applied to the solution of Navier-Stokes equations in finite elements

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Abstract

This paper deals with the numerical solution of 3-dimensional Navier-Stokes equations with a free surface, in the framework of finite elements and the Telemac system. The idea of the wave equation is applied here. It was previously used for solving Shallow Water Equations and consists of eliminating the velocity in the depth-averaged continuity equation thanks to an expression obtained with the momentum equations. After an overview of the basic solution procedure in Telemac-3D we detail the new algorithm and show its advantages in terms of computer time.
Keywords: wave equation, pseudo-wave equation, Navier-Stokes equations, finite elements, Telemac system.

1 Introduction

The hydroinformatic system Telemac, based on Finite Element techniques, addresses free surface and underground flows. In its present development stage, the system includes the Saint-Venant or shallow water equations (Telemac-2D), Navier-Stokes equations in 3 dimensions with a free surface (Telemac-3D), and also mild slope equations, wave action equations, water quality models, sediment transport equations in 2D and 3D, Richard's equations in 2D and 3D.

Recent advances have led to a robust algorithm for the treatment of full non-hydrostatic 3D Navier-Stokes equations with a free surface, which allows rapid flows, hydraulic jumps, wetting and drying, so that computations of a dam-break flood waves can now be performed (see ref. [7]).

The solution procedure in Telemac-3D is based upon a fractional step approach and, in previous versions, one of the steps was a solution of Shallow

Water Equations. This step included a treatment of the free surface gradients and the computation of the new depth. We now suppress this step and use instead the idea of the wave equation, consisting of eliminating the velocities in the depth-averaged continuity equation, which gives a wave equation with the depth as only unknown. In the next section, we briefly describe all the fractional steps of our basic algorithm. Then in section 3 we detail the wave equation technique.

2 Description of the basic solution procedure

We want to solve the Navier-Stokes, with or without hydrostatic assumption, and written in the form:

Continuity:

$$div(\vec{U}) = 0 \tag{1}$$

Momentum:

$$\frac{\partial U}{\partial t} + \vec{U}.\overrightarrow{grad}(U) = -\frac{1}{\rho_0}\frac{\partial p}{\partial x} + div(\nu_t \overrightarrow{grad}(U)) + f_x \tag{2}$$

$$\frac{\partial V}{\partial t} + \vec{U}.\overrightarrow{grad}(V) = -\frac{1}{\rho_0}\frac{\partial p}{\partial y} + div(\nu_t \overrightarrow{grad}(V)) + f_y \tag{3}$$

$$\frac{\partial W}{\partial t} + \vec{U}.\overrightarrow{grad}(W) = -\frac{1}{\rho_0}\frac{\partial p}{\partial z} + div(\nu_t \overrightarrow{grad}(W)) + f_z \tag{4}$$

Where p is the pressure, ρ_0 is the reference density, ν_t the turbulent viscosity, and f_x and f_y are source terms such as the Coriolis force and buoyancy. U, V, and W are the Cartesian velocity components of the velocity $\vec{U}$. The friction will appear later as boundary condition of the diffusion terms. With a hydrostatic pressure assumption, equation (4) is discarded.

The basic option in Telemac-3D consisted of resorting to fractional steps:

1) an advection step solving:

$$\frac{\vec{U}^C - \vec{U}^n}{\Delta t} + \vec{U}.\nabla(\vec{U}) = 0 \tag{5}$$

Δt is the time-step, $\vec{U}^n$ the velocity field at the old time-step, and $\vec{U}^C$ the result of this advection step. Here the method of characteristics, the S.U.P.G. method, or distributive schemes can be used.

2) a diffusion and source terms step solving:

$$\frac{\vec{U}^{D} - \vec{U}^{C}}{\Delta t} = div(\nu_t grad(U)) + f_x \qquad (6)$$

$$\frac{\vec{V}^{D} - \vec{V}^{C}}{\Delta t} = div(\nu_t grad(V)) + f_y \qquad (7)$$

3) then the pressure is split into the hydrostatic pressure p_h and dynamic pressure π. We have $p_h = \rho g(Z_s - z) + \rho g \int_z^{Z_s} \frac{\Delta\rho}{\rho} dz$, where ρ is the density, g the gravity acceleration, Z_s is the free surface elevation and the second term takes into account the buoyancy effects due to temperature or salinity. $\Delta\rho$ is ρ-ρ_0. The free surface gradients appear in the momentum equations, and they are treated along with a depth-averaged continuity equation:

$$\frac{U^{hyd1} - U_D}{\Delta t} = -g\frac{\partial Z_s}{\partial x} \quad \text{and} \quad \frac{V^{hyd} - V_D}{\Delta t} = -g\frac{\partial Z_s}{\partial y} \qquad (8)$$

$$\frac{\partial h}{\partial t} + div(h\vec{u}) = 0 \qquad (9)$$

where U^{hyd} and V^{hyd} are the final velocities of the hydrostatic option and $\vec{u}$ is the depth-averaged velocity field. If we depth-average these equations, it yields:

$$\frac{u^{n+1} - \overline{U_D}}{\Delta t} = -g\frac{\partial Z_s}{\partial x} \quad \text{and} \quad \frac{v^{n+1} - \overline{V_D}}{\Delta t} = -g\frac{\partial Z_s}{\partial y} \qquad (10)$$

where u^{n+1} and v^{n+1} are the final depth averaged velocities and the bar expresses the depth-average operator. This system can be considered as Shallow Water momentum equations without advection, friction nor diffusion, and can be solved by a Shallow Water Equations solver as Telemac-2D. The depth averaged continuity equation is also solved concurrently, and yields a new free surface. This equation is obtained without any approximation and is thus also valid in a 3D context. The 3D velocities are then easily retrieved because:

$$\frac{U^{hyd} - U_D}{\Delta t} = \frac{u^{n+1} - \overline{U_D}}{\Delta t} \quad \text{and} \quad \frac{V^{hyd} - V_D}{\Delta t} = \frac{v^{n+1} - \overline{V_D}}{\Delta t} \qquad (11)$$

4) At this stage, hydrostatic and non-hydrostatic options have a different treatment.

With a hydrostatic option, U^{hyd} and V^{hyd} are the final horizontal velocities and the final vertical velocity W is obtained from the 3-dimensional continuity equation.

With a non-hydrostatic option, an equation for W is added:

$$\frac{\partial W}{\partial t} + \vec{U}.grad(W) = -\frac{1}{\rho_0}\frac{\partial \pi}{\partial z} + div(\nu_t grad(W)) \qquad (12)$$

and the hydrodynamic pressure is taken into account to ensure the continuity equation and yield a divergence free velocity field. Namely, taking U^{hyd} and V^{hyd} from the hydrostatic option, we now solve:

$$\frac{\vec{U}^{n+1} - \vec{U}^{hyd}}{\Delta t} = -\frac{1}{\rho_0}\overrightarrow{grad}(\pi) \qquad (13)$$

and given the fact that $div(\vec{U}^{n+1}) = 0$, it yields:

$$div(\vec{U}^{hyd}) = div(\frac{\Delta t}{\rho_0}\overrightarrow{grad}(\pi)) \qquad (14)$$

which is a Poisson equation. Once this equation is solved, the final velocity field is given by equation (13). Finally, the free surface is updated again with the help of the depth-averaged continuity equation, where the velocity is known and the depth h^{n+1} at time t^{n+1} is the unknown.

3 The wave equation

The new procedure consists of replacing the steps 2 and 3 by the following algorithm. Here we assume that there is a hydrostatic pressure assumption and will use the notation U^{n+1} and V^{n+1} instead of U^{hyd} and V^{hyd}. However step 4 can be added. We now want to solve the depth-averaged continuity equation discretised in the form:

$$\frac{h^{n+1} - h^n}{\Delta t} + div(h\left[\theta_u \vec{u}^{\,n+1} + (1-\theta_u)\vec{u}^{\,n}\right]) = 0 \qquad (15)$$

where θ_u is an implicitation coefficient for the velocity.

The (hydrostatic) momentum equation is written in the form:

$$\frac{\vec{U}^{n+1} - \vec{U}^{C}}{\Delta t} = -g\theta_h \overrightarrow{grad}(h^{n+1} - h^n) - g\,\overrightarrow{grad}(Z_s^n) + div(\nu_t \overrightarrow{grad}(\vec{U})) + \vec{f} \qquad (16)$$

where θ_h is an implicitation coefficient for the depth. The goal is to get an expression of the velocity which will have the depth as only unknown. This is obtained after the variational formulation by mass-lumping the derivative in time and by resorting to an explicit form of the diffusion. However the friction is kept implicit. As a matter of fact, we write:

$$\int_{\Omega 3D} div(\nu_t\, \overrightarrow{grad}(\vec{U}))\ \Psi_i^{3D}\ d(\Omega 3D) \approx$$

$$\begin{aligned} &\int_{\Gamma 3D} \Psi_i^{3D}\ \nu_t\ \overrightarrow{grad}(\vec{U}^{n+1}).\vec{n}\ \ d(\Gamma 3D) \\ &-\int_{\Omega 3D} \nu_t\ \overrightarrow{grad}(\vec{U}^{n})\ \overrightarrow{grad}(\Psi_i^{3D})\ d(\Omega 3D) \end{aligned} \tag{17}$$

$\Omega 3D$ is the three-dimensional domain, $\Omega 2D$ will be the two-dimensional domain, projection of the former on the vertical. The 3D mesh is obtained by a superimposition of 2D meshes of triangles. Ψ_i^{3D} are the test functions in the 3D mesh, Ψ_i^{2D} will be the test functions in the 2D mesh. Note that in equation (17) the boundary terms are implicit. Our diffusion terms are thus such that we can write:

$$\frac{\int_{\Omega 3D} div(\nu_t\, \overrightarrow{grad}(\vec{U}))\ \Psi_i^{3D}\ d(\Omega 3D)}{\int_{\Omega 3D} \Psi_i^{3D}\ \ d(\Omega 3D)} = -\ frot3d\ \ \vec{U}^{n+1} - \overrightarrow{DIFF} \tag{18}$$

with:

$$\overrightarrow{DIFF} = \frac{\int_{\Omega 3D} \nu_t\, \overrightarrow{grad}(\vec{U}^{n})\ \ \overrightarrow{grad}(\Psi_i^{3D})\ d(\Omega 3D)}{\int_{\Omega 3D} \Psi_i^{3D}\ \ d(\Omega 3D)} \tag{19}$$

and defining *frot3d* by:

$$\frac{\int_{\Gamma 3D} \Psi_i^{3D}\ \nu_t\ grad(\vec{U}^{n+1}).\vec{n}\ \ d(\Gamma 3D)}{\int_{\Omega 3D} \Psi_i^{3D}\ \ d(\Omega 3D)} = -\ frot3d\ \vec{U}^{n+1} \tag{20}$$

The final expression of the velocity will be:

$$\vec{U}^{n+1} = \frac{\vec{U}^{C} + \Delta t[-g\theta_h\ \overrightarrow{grad}(h^{n+1} - h^n) - g\ \overrightarrow{grad}(Z_s^n) + \vec{f} - \overrightarrow{DIFF}]}{1 + \Delta t\ frot3d} \tag{21}$$

We now detail the discretisation of the continuity equation. In the variational formulation, the divergence term is integrated by part, which gives:

$$\begin{aligned} &\int_{\Omega 2D} \Psi_i^{2D} (\frac{h^{n+1} - h^n}{\Delta t}) d(\Omega 2D) + FLUX2D \\ &-\int_{\Omega 2D} h\left[\theta_u \vec{u}^{n+1} + (1-\theta_u)\vec{u}^{n}\right].\overrightarrow{grad}(\Psi_i^{2D}) d(\Omega 2D) = 0 \end{aligned} \tag{22}$$

where:

$$FLUX2D = \int_{\Gamma 2D} \Psi_i^{2D} h\left[\theta_u \vec{u}^{n+1} + (1-\theta_u)\vec{u}^{n}\right].\vec{n}_{2D}\ d(\Gamma 2D) \tag{23}$$

$\Gamma 2D$ is the boundary of the 2D mesh.

FLUX2D can be obtained by summing the following terms on the vertical:

$$FLUX3D = \int_{\Gamma 3D} \Psi_i^{3D} \left[\theta_u \vec{U}^{\,n+1} + (1-\theta_u)\vec{U}^{\,n} \right] .\vec{n}_{3D}\, d(\Gamma 3D) \tag{24}$$

At this level, impermeability of solid boundaries can be obtained in a weak form in the sense of finite elements by simply cancelling the corresponding fluxes through walls.

Depth-averaged velocities are then replaced by their definition:

$$\int_{\Omega 2D} \Psi_i^{2D} \left(\frac{h^{n+1} - h^n}{\Delta t}\right) d(\Omega 2D) + FLUX2D -$$

$$\int_{\Omega 2D} \left[\theta_u \int_{Z_f}^{Z_s} \vec{U}^{\,n+1} dz + (1-\theta_u) \int_{Z_f}^{Z_s} \vec{U}^{\,n} dz \right] .\overrightarrow{grad}(\Psi_i^{2D})\, d(\Omega 2D) = 0 \tag{25}$$

where the integral on the vertical can be obtained by the trapezoidal rule. We now use our expression of the velocity into the continuity equation, to get:

$$\int_{\Omega 2D} \Psi_i^{2D} \left(\frac{h^{n+1} - h^n}{\Delta t}\right) d(\Omega 2D) +$$

$$\int_{\Omega 2D} \left[\int_{Z_f}^{Z_s} \frac{1}{\frac{1}{\Delta t} + frot3d} dz \right] \left[g\theta_h\, \theta_u\, \overrightarrow{grad}(h^{n+1} - h^n) \right] .\overrightarrow{grad}(\Psi_i^{2D})\, d(\Omega 2D)$$

$$= \int_{\Omega 3D} \left[\theta_u \vec{U}^{\,aux} + (1-\theta_u)\vec{U}^{\,n} \right] .\overrightarrow{grad}(\Psi_i^{2D}) d(\Omega 3D) - FLUX2D \tag{26}$$

with the notation:

$$\vec{U}^{\,aux} = \frac{\vec{U}^{\,C} + \Delta t(\vec{f} - \overrightarrow{DIFF} - g\, \overrightarrow{grad}(Z_s^n))}{1 + \Delta t\ frot3d} \tag{27}$$

At the second line of equation (26), we recognize diffusion terms, the role of the diffusion coefficient being played by:

$$\nu_{wave} = g\, \theta_h\, \theta_u \int_{Z_f}^{Z_s} \frac{1}{\frac{1}{\Delta t} + frot3d} dz \tag{28}$$

This formal diffusion accounts for the propagation of shallow water waves at the celerity $\sqrt{gh}$, hence the name wave equation though we have here formally a diffusion equation. The fact that we get here a diffusion equation rather than a wave equation is an artifact due to the finite difference treatment of the derivative in time in the momentum equation. As a matter of fact, ν_{wave} is not in m2/s, it is the time step Δt multiplied by the square of a celerity, and dividing all equation (26) by Δt would indeed give a wave equation.

The unknown in the equation (26) is h^{n+1}-h^n, the matrix is a simple and symmetric stiffness matrix, which is easily inverted by the conjugate gradient method.

When the new depth h^{n+1} is known, the velocity field is simply retrieved by computing:

$$\overrightarrow{U}^{n+1} = \overrightarrow{U}^{aux} + \Delta t \frac{\left[\frac{\int_{\Omega 2D}[-g\theta_h \overrightarrow{grad}(h^{n+1}-h^n)]\ \Psi_i^{2D}\ d(\Omega 2D)}{\int_{\Omega 2D}\Psi_i^{2D}\ d(\Omega 2D)}\right]}{1+\Delta t\ frot3d} \tag{29}$$

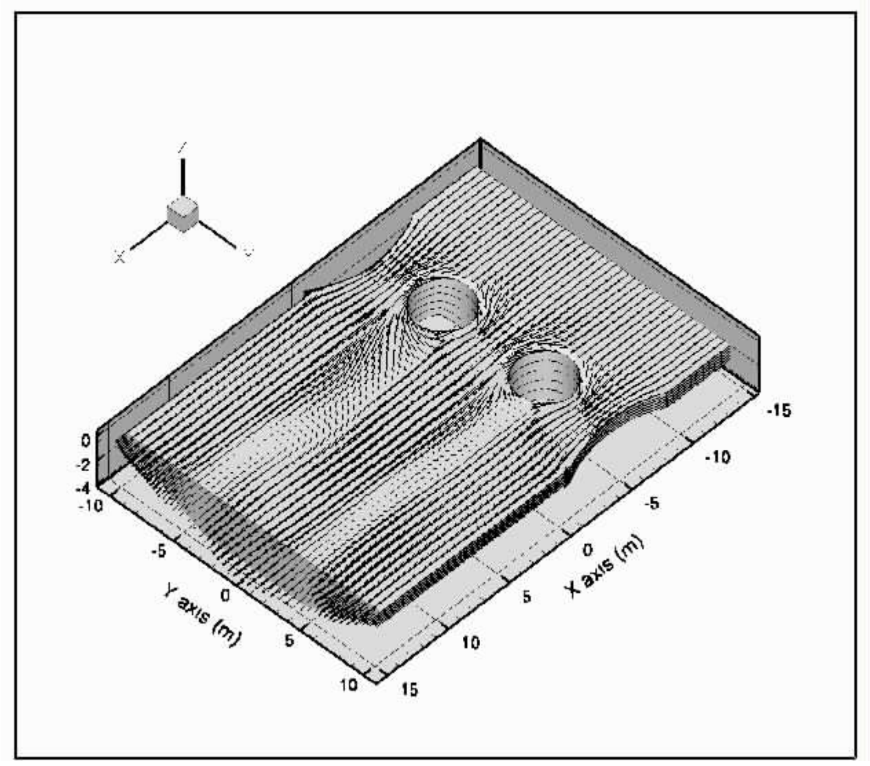

Figure 1: Flow around bridge piers, showing von Karman eddies.

4 Results

We present here the example of a quasi-steady flow around bridge piers in a river. Figure 1 shows a view of the domain with the velocities at the free surface, obtained with the non-hydrostatic option. A constant eddy viscosity set to 0,005 m^2/s enables the formation of von Karman eddies.

The simulation consists of 800 time steps of 0.1 s each. The mesh is made of 5 layers of prisms obtained with the superimposition of 6 2-dimensional meshes of 4304 triangles. We have thus 21520 prisms in the 3D mesh.

The computer times on a 750 MHz Unix machine are given in table 1 below: The original algorithm described in section 2 is called “Shallow Water Equations”

Table 1: Computer times of Telemac-3D on the bridge piers case.

	Step 3 with Shallow Water Equations	Step 2 + step 3 with Wave equation
Hydrostatic assumption	261 s	129 s
Non-hydrostatic	695 s	516 s

The computer time is divided by 2 in the case of a hydrostatic assumption. With the full Navier-Stokes equations, the effect is less dramatic, as the projection step for ensuring a divergence free velocity is the most time-consuming part of the algorithm. However the time is reduced by 179 s, which is greater than the reduction of 132 s obtained with the hydrostatic assumption. This can be explained by the fact that the velocity divergence is less affected by the wave equation than the previous algorithm.

A weakness of the method is the fact that the diffusion terms are treated in an explicit way. It is a key point that in some cases forbids large time steps. To cope with this problem, one can however treat implicitly the diagonal terms of the diffusion matrix. In this case, the diffusion terms in the momentum equations:

$$\overleftrightarrow{U}_i^{n} \frac{\int_{\Omega 3D} \nu_t \, \overleftrightarrow{grad}(\Psi_i^{3D}) \, \overleftrightarrow{grad}(\Psi_i^{3D}) \, d(\Omega 3D)}{\int_{\Omega 3D} \Psi_i^{3D} \; d(\Omega 3D)} \tag{30}$$

are replaced by:

$$\vec{U}_i^{n+1} \frac{\int_{\Omega 3D} \nu_t \, \overrightarrow{grad}(\Psi_i^{3D}) \, \overrightarrow{grad}(\Psi_i^{3D}) \, d(\Omega 3D)}{\int_{\Omega 3D} \Psi_i^{3D} \; d(\Omega 3D)} \tag{31}$$

This creates new implicit terms which add in the formulation, without extra complexity, to the friction terms *frot3d*. A significant enhancement of stability is then obtained.

5 Conclusion

The method presented here is faster than the algorithm which was previously used in Telemac-3D and takes advantage of techniques which were already successfully employed in Shallow Water Equations, such as the wave equation method and the weak formulation of impermeability. Robust 3D non-hydrostatic Navier-Stokes solvers with a free surface are now available for a wide range of applications including waves, wetting and drying, density effects, supercritical flows and hydraulic jumps. Further improvements are still necessary to deal with strong stratifications, and the efforts are now put on the sigma transformation which could be suppressed, but is still used, at least in the advection step. Unstructured meshes on the vertical, e.g. with tetrahedrons, would also allow a better local treatment of outfalls. However depth-averaging would raise problems on a fully unstructured mesh. Time will tell if viable algorithms can be developed in finite elements, or if brand new techniques such as Smooth Particle Hydrodynamics will take the lead.

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