A fully nonlinear Boussinesq model for surf zone hydrodynamics

R. E. Musumeci¹, I. A. Svendsen², J. Veeramony³ & E. Foti¹
¹Department of Civil and Environmental Engineering, Catania University, Italy.
²Center for Applied Coastal Research, University of Delaware, U.S.A.
³MSU ERC at Stennis, Stennis Space Center (U.S.A.)

Abstract

This work aims to represent the dynamics of the wave propagation in the surf zone through a weakly dispersive fully nonlinear Boussinesq model. Since the breaking process introduces a huge amount of vorticity within the flow, the usually adopted hypothesis of potential flow has been removed here, and thus the momentum increase due to breaking has been modeled directly in terms of the vorticity field. The amount of vorticity introduced by the breaking process has been determined through an analogy with the hydraulic jump and the adoption of the concept of the roller. An accurate description of the effects of the surface roller has been reached using an original self-adaptive-time-varying grid, developed on purpose, this approach allowing to get a better resolution just in the region where the vorticity is generated. Comparisons with the version of the model using a fixed uniform computational grid showed that the new approach allows to better recover the impulsive inclusion of vorticity within the flow and in turn the energy dissipation within the surf zone. Comparisons with laboratory measurements, both for regular and irregular waves, demonstrated that the proposed model has fairly good prediction capabilities.
1 Introduction

Due to the interaction with a sloping bottom the waves shoal, thus increasing the height, the asymmetry and the skewness of the wave shape. When the modified wave characteristics cannot be anymore sustained, the waves are forced to break, and the wave energy associated to the organized wave motion is thus transformed into turbulence. This high dissipative process strongly affects the generation of longshore and cross-shore currents. The mechanism of vorticity generation near the free surface is still to be established firmly, but it seems to be clear that the steepening of the wave naturally induces high curvature and then vorticity inside the flow. In particular, the vorticity has a maximum in the region of intense shear immediately down stream of the separation of the free surface. After its generation from the source region on the front of the wave, close to the surface roller region \[1,2\], the vorticity flux is convected downward due to viscous effects. A sketch of the roller region is shown in Figure 1. It is worth to remind that the roller is the recirculating part of the flow above the dividing streamlines, being located on the front of a breaking wave and moving with approximately the same speed of the wave. An understanding and correct modeling of the aforementioned processes is needed in the framework of large scale models aimed to predict not only wave propagation, but also the morphological evolution of the nearshore region.

Depth integrated models, such as nonlinear shallow water equations and Boussinesq equations, are often used in order to model the wave motion in shallow waters. The nonlinear shallow water equations are particularly suitable in very shallow waters and non-breaking waves, e.g. where the horizontal velocity profile can be assumed to remain constant over depth. These models have been largely used to investigate phenomena such as the run-up of both regular and irregular waves \[3, 4\]. The Boussinesq equations allow to get a more realistic representation of the velocity profiles and of the wave front, and thus have been extended to be used more offshore, in deeper waters \[5, 6, 7\], then improving their applicability to more practical situations.

![Figure 1: Roller of a breaking wave.](image)

Here the description of the flow through the surf zone is pursued through a fully nonlinear weakly dispersive Boussinesq model. By removing the traditional unrealistic hypothesis of irrotational motion, the presented model allows to consider the injection of vorticity due to the breaking phenomenon, which strongly affects the velocity distribution over the depth and influences the energy dissipation within the surf zone.
In the proposed model, originally presented in a weak nonlinear version in [8] and then in a fully nonlinear version in [9], which has been updated and debugged, the roller is modeled as the source of vorticity for the flow.

2 Mathematical formulation

The wave propagation is here modeled by using a 1D fully nonlinear Boussinesq-type model. The main novelty of the model is that as the breaking process causes a very strong vortical motion at different scales, in order to model appropriately breaking waves inside the surf zone, the flow must be considered rotational. For gentle sloping beaches and spilling breaker waves and neglecting the effects of the bottom boundary layer, under the hypotheses of incompressible fluid, impermeable and fixed bed, the governing dimensionless equations can be expressed in terms of two dimensionless parameters: (i) $\mu$, the dispersive parameter, i.e. the ratio between the local water depth and the actual wavelength, and (ii) $\delta$, the nonlinear parameter, i.e. the ratio between the wave amplitude and the water depth.

In order to specify the assumed velocity profile, it has been considered the following expression of the vorticity

$$\omega = \mu^2 \psi_{xx} + \psi_{zz}$$

in terms of the stream function $\psi$. By integrating over depth eq. (1) and by considering that at the bottom the stream function must be zero, it turns out that the horizontal velocity is expressed as follows

$$u(x, y, z, t, \omega) = u_p(x, y, z, t) + u_r(x, y, z, t, \omega)$$

where $u_p$ is the potential velocity and $u_r$ is the rotational velocity, which expresses the dependence of the velocity from the vorticity field.

Figure 2: Schematization of the problem and adopted reference system.
The following continuity and momentum equations are then derived by assuming the reference system as shown in Figure 2, the dependent variables being the surface elevation $\zeta$ and the depth integrated velocity $\bar{u}$:

$$\zeta_t + \left([h + \delta \zeta] \bar{u}\right)_x = 0$$  \hspace{1cm} (3)

$$\bar{u}_t + \delta \bar{u}_x + \zeta_x + \mu^2 \left[\left(B - \frac{1}{3}\right) h^2 \bar{u}_{xx} - \frac{1}{2} h \bar{u}_x \bar{u}_t - h \bar{u}_x \bar{u}_t \right] + B \mu^2 h^2 \bar{u}_{xx}$$

$$+ \delta \mu^2 \left[\frac{1}{3} h^2 \bar{u}_{xxx} - h \zeta_x \bar{u}_x + \frac{1}{3} h^2 \bar{u}_x \bar{u}_{xx} - \frac{2}{3} h \bar{u}_x \bar{u}_t - \frac{3}{2} h \bar{u}_x \bar{u}_x \right]$$

$$- \frac{1}{2} h \bar{u}_x \bar{u}_xx - h \bar{u}_x \bar{u}_x - \delta \bar{u}_x \bar{u}_t - \frac{1}{2} \delta \bar{u}_x \bar{u}_t + B h^2 \bar{u}_x$$

$$+ \delta^2 \mu^2 \left[\frac{1}{6} \delta \zeta \bar{u}_x - \frac{1}{3} \delta \bar{u}_x \bar{u}_x \bar{u}_t - \frac{1}{3} h \bar{u}_x \bar{u}_x \bar{u}_t + h \bar{u}_x \bar{u}_t - \frac{1}{2} \delta \bar{u}_x \bar{u}_x \right]$$

$$- \frac{2}{3} h \bar{u}_x \bar{u}_x \bar{u}_x - \delta \bar{u}_x \bar{u}_x - \delta \bar{u}_x \bar{u}_x - \frac{1}{2} \delta \bar{u}_x \bar{u}_x \bar{u}_x - \frac{3}{2} \delta \bar{u}_x \bar{u}_x \bar{u}_x - \delta \bar{u}_x \bar{u}_x$$

$$+ \delta^3 \mu^2 \left[\delta \zeta \bar{u}_x - \delta \zeta \bar{u}_x \bar{u}_x + \delta \zeta \bar{u}_x \bar{u}_x \right]$$

$$- \frac{1}{3} \delta \zeta \bar{u}_x \bar{u}_x \bar{u}_x + \delta \zeta \bar{u}_x \bar{u}_x \bar{u}_x - \frac{1}{3} \delta \zeta \bar{u}_x \bar{u}_x \bar{u}_x$$

$$+ \delta^2 \mu^2 \left[\delta \zeta \bar{u}_x \bar{u}_x \bar{u}_x + \delta \zeta \bar{u}_x \bar{u}_x \bar{u}_x \right]$$

$$= O(\mu^4)$$  \hspace{1cm} (4)

$B$ is the parameter introduced by [5] in order to improve the dispersive characteristic of the model. Eq. (3) represents the exact expression of the continuity equation, while the momentum equation (4) is approximated up to $O(\mu^4)$ while it is exact in the nonlinear parameter $\delta$. The model is then fully nonlinear and it allows to model waves close to breaking, i.e. highly nonlinear waves, on the contrary of the standard Boussinesq model, where the dispersive effects are exactly balanced by the nonlinear effects and the wave steepness cannot increase realistically. Moreover, as the expression of the velocity $\bar{u}$ depends on the vorticity $\omega$, terms representing the effects of the vorticity on the field appear within eq. (4), namely $(\Delta M)_x + (\Delta P)_xx, (\Delta M)_x D, e D_{vv}$, expressing the different contributions for the increase of momentum flux after breaking. It is worth pointing out that these terms have been derived directly as a consequence of eq. (2).

In order to solve eqs. (3) and (4), one more equation, i.e. the vorticity transport equation, need to be taken into account

$$\omega_t + \delta \bar{u}_x \omega_x + \delta \bar{v}_y \omega_z = \nu_t (\mu^2 \omega_{xx} + \omega_{zz})$$  \hspace{1cm} (5)

to determine the vorticity field. The boundary conditions of eq. (5) express the fact that at the bottom the vorticity must be zero and so it has to be on the free
surface, everywhere but on the front of the wave, that is corresponding to the roller region (see Figure 2). Indeed, the vorticity field is not solved within the roller, but the value \( \omega_s(x, t) \) it is assumed at the lower edge of it. In formula, this reads as

\[
\omega(x, z = \zeta_e, t) = \omega_s(x, t)
\]

(6)

where \( \zeta_e \) is the elevation of the lower edge of the roller. Then eq. (5) is solved analytically by means of a perturbation approach.

3 Modelling of the breaking process

It is well known that the Boussinesq models are not able to start the breaking, therefore an external empirical breaking criterion need to be adopted in order to decide where and how the breaking starts. Here a criterion based on the critical wave slope has been adopted \[10\], in order to determine the geometry of the surface roller.

According to eq. (4) the breaking process is modeled through the aforementioned breaking terms, which are closely related to the vorticity field solved by eq. (5), forced by the boundary condition (6), which represents the source of vorticity. In order to specify \( \omega_s \) in \[9\] the analogy between the hydraulic jump and the surface roller of a breaking wave has been proposed to specify the vorticity at the lower edge of the roller. In particular, a best fit of experimental data on hydraulic jumps, with Froude number similar to those of surf zone waves (within 1.38 \( \pm \) 1.56), has been used \[11\]. In dimensionless form this reads to:

\[
\omega_s = 15.75 \left( 1 - \frac{x - x_t}{l_r} \right)
\]

(7)

where \( x_t \) is the position of the toe of the roller and \( l_r \) is the length of the roller (see Figure 3b). In order to avoid numerical instabilities due to the sharp discontinuity at the toe of the roller, where the biggest inclusion of vorticity takes place, eq. (7), described by the solid line in Figure 3a, has been approximated at the toe, thus smoothing out the transition region (see dashed line in Figure 3a).

In the present model the same approach has been adopted. Moreover a self-adaptive time-varying grid has been introduced in order to avoid the losses of vorticity induced by the uniform fixed grid adopted to solve the Boussinesq equations. The proposed subgrid moves in time and space adapting its definition to the local position of the roller, being finer only close to the toe (see Figure 3b).

The effects of using such a grid are evident in Figure 4, where the vorticity distributions obtained by using the traditional uniform fixed grid (see Figure 4a)
and the new moving grid (see Figure 4b) are compared. It is clear that the new grid allows to get a bigger amount of vorticity inside the flow. Moreover, as a reference, in Figure 4 the experimental surface profile, recovered by [12], is reported, showing that the subgrid allows to get a more realistic picture of the flow.

Figure 3: a) Vorticity distribution at the lower edge of the roller (solid line fit of data from [11]; dashed line approximated curve as used in the numerical model) b) scheme of the fixed grid (circles) and of the proposed self adaptive time-varying grid (dots) under the roller.

4 Surf zone hydrodynamics: comparisons with data

The model results have been compared with experimental data obtained from literature both for regular and irregular waves. In particular, in Figure 5a the comparisons of the model results with the wave height measurements of [13] obtained for three different wave conditions are shown. The good shoaling properties of the model appear clearly, indeed only at the breaking point the wave height is slightly underestimate. Inside the breaking region the moving grid model allows to reproduce better the decrease of wave height compared to the previous version of the model. The error on the wave height prediction is reduced from about 26% to 14%. Also the case of plunging breakers (see Figure 5b), which is outside of the theoretical model capabilities, is reproduced quite reasonably.

Hansen and Svendsen [13] provided also the values of wave speed, which are extremely important to study wave propagation. Figure 5b shows the comparisons of the data with the model results, which result quite good both for spilling and plunging breakers. In the latter case, moreover, the same scatter which appears in the data is reproduced by the numerical results.

Another characteristic of the presented model, which is not common for other Boussinesq models, is its capability to calculate the undertow. The undertow is an offshore-going current that is generated by the increase of volume flux carried forward by the surface roller of a breaking waves. Since this model takes into account the effects of the roller, it is possible to evaluate the undertow (see Figure 6). The comparisons with the experimental data of [14] are obviously lack
near the bottom because of the free slip condition. The fixed grid model seems to provide better results, however it should be mentioned here that the eddy viscosity model used in both cases has been assumed depth invariant, for sake of simplicity. When the subgrid model is used, then, due to the increase of vorticity, lots of turbulence is transported downward, while measurements have shown that under breaking waves the turbulence is confined in the upper part of the fluid.

![Figure 4: Vorticity distribution and surface profile: results obtained using a) the uniform fixed grid and b) the self adaptive time varying grid (dashed line: measured surface profile, from [12]).](image)

Finally, the model behaviour, when the subgrid is adopted, has been tested against the experimental data on breaking groupy waves of [15], which represents a simplified case of irregular waves. The comparison with the time series of the surface profiles, shown in Figure 7, is fairly good, both in the shoaling and in the surf zone. Indeed, it can be noticed that, both in the measurements and in the numerical results the groupiness is not destroyed but conserved during the breaking process and that exchanges of energy between different frequency take place. Moreover, in [15] the authors noticed a change of the position of the highest waves of the group, related to the others. Indeed it seemed that on the slope, higher height waves travel faster than the smaller ones, changing also the period and the wave length. This is more evident when the groupiness is higher.
Figure 5: a) Wave height and b) Wave speed: diamonds measured data [13]; solid line moving grid model results; dashed line fixed grid model results: (a) $T=2.5 \, s \, H_0=0.040 \, m$; (b) $T=3.33 \, s \, H_0=0.042 \, m$.

Figure 6: Undertow: dots comparisons with data from [14], solid line moving grid model results; dashed line fixed grid model results.

The proposed model recovers some features pretty well, such as the changes of wave height distribution inside the group and the period and length variation of the individual waves. Moreover when the groupiness is large, another phenomenon which can be noticed both in the data and in the numerical results is the bore-bore capturing process, which occurs in the inner surf zone, far from the breaking point. This process is also often seen on natural beaches. As an effect of the wavelength variations and of the bore-bore capturing process, the number of waves within the domain is reduced in the inner surf zone.

Since the groups were generated packing together five different cnoidal waves, the distribution of wave height along the x-axis was compared with the data for each component of the wave group. The agreement was quite satisfactory both in the shoaling region and in the inner surf zone, even though the comparison is not reported for the sake of space. In the transition region, there is scatter in the...
experimental data and the agreement with model results is reasonable good everywhere, though seems to be much better for the highest waves.

Figure 7: Comparisons with wave group data [15]. Time series of the surface profile (dashed line data, solid line moving grid model results).

5 Conclusions

A new approach to the modeling of the breaking process implemented into a 1D fully nonlinear Boussinesq model has been proposed here. In particular the breaking induced dissipation has been modeled through breaking terms in the momentum equation, which are derived directly from the expression of the velocity, which, in the case being is also function of the vorticity field. From a numerical viewpoint, a self-adaptive time-varying grid has been implemented in order to recover the sudden increase of vorticity due to passage of the surface roller.

The comparisons with literature laboratory data, both for regular and groupy waves, show the quite good predictive capabilities of the model (e.g. the errors on the wave height within the surf zone is reduced from 26% to 14% when the subgrid is adopted). Even though the model is theoretically able also to reproduce the undertow profile, the depth invariant eddy viscosity adopted seems to negatively affect the comparisons with laboratory data. The testing of the model against almost irregular waves, such wave groups, shows that the model is also able to reproduce physical phenomena, such as the frequency and energetic exchanges within the group or the bore-bore capturing process.
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References