Numerical modelling of shallow water using an iterative solution algorithm

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Abstract

The knowledge of the current patterns and surface elevations of lakes and reservoirs is an essential need for a number of engineering problems such as water quality, flooding, sediment transport and coastal erosion. Furthermore, the design of coastal and offshore structures requires information on currents for the determination of the forces affecting these structures. In this study, a numerical model has been developed for simulating the current patterns in a well-mixed lake. The hydrodynamic simulation is based on solving the shallow-water equations using the Newton-Raphson algorithm. The results have been compared to Leendertse's multi-operational finite difference scheme. Developed model can be used efficiently for long-term current patterns and real-time applications.

1 Introduction

Knowing velocities and water level changes in shallow water bodies is needed by various areas of civil and environmental engineering. Current patterns and depth changes should be known for the purposes like the design of coastal and sea structures, thermal pollution, coastal sediment transportation, determination and conservation of water quality, fisheries and recreation. As the analytical solutions for the momentum and continuity equations expressing the velocity and depth changes in water bodies cannot be found various numerical solution methods and calculation schemes like finite differences, finite elements, characteristics and spectral methods are developed.

In most of the present numerical models velocity components and depths are calculated by sliding the time-varying, non-linear partial differential equation system over time and space for linearisation. During this operation; a quantity
like \([f(t)]^2\) is written in the form \(f(t-\Delta t)\ast f(t)\) for linearisation ([1]-[4]). This assumption gives reliable results when \(\Delta t\) time steps are chosen small enough. In explicit models the solution can only be made when the Courant number is less than 1. In the implicit models when the Courant number is less than or equal to 5 the obtained results don't change much but for the bigger values the reliability of the results decrease while the stability is secured.

In the presented study, because of all the quantities are written in the same time step using an iterative solution algorithm beginning with the initial values, the time step \(\Delta t\) between two consequent solutions can be chosen large and the solution can be made for higher Courant numbers. So the computation time decreases, the usage of the model for the solution of problems especially about water quality needing long computation times becomes possible and sensibility of the results increase. The model results are compared with the solution methods given in reference [5].

2 Basic equations

The continuity and momentum equations in the well mixed coastal zones where there is no stratification and the vertical water movements can be neglected by horizontal water movements, may be made two dimensional in space using average values calculated over depth ([4], [6]).

Momentum equation in \(x\)-direction:

\[
\frac{\partial U}{\partial t} + U \frac{\partial U}{\partial x} + V \frac{\partial U}{\partial y} - fV + \frac{\partial}{\partial x} \left( \frac{\partial \eta}{\partial x} + \frac{g}{c^2} \frac{U^2 + V^2}{C^2} \right) - A \left( \frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} \right) - F_x = 0
\]  

(1)

Momentum equation in \(y\)-direction:

\[
\frac{\partial V}{\partial t} + U \frac{\partial V}{\partial x} + V \frac{\partial V}{\partial y} + fU + \frac{\partial}{\partial x} \left( \frac{\partial \eta}{\partial y} + \frac{g}{c^2} \frac{U^2 + V^2}{C^2} \right) - A \left( \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} \right) - F_y = 0
\]  

(2)

Continuity equation:

\[
\frac{\partial \eta}{\partial t} + \frac{\partial (U \eta)}{\partial x} + \frac{\partial (V \eta)}{\partial y} = 0
\]  

(3)

can be written.

In these equations:

\(u, v\) : vertically averaged velocities in \(x\) and \(y\)-directions
\(\eta\) : free water depth according to average lake surface
\(D\) : water depth according to average lake surface
\(H\) : total water depth \(d + \eta\)
\[ \begin{align*} f &: \text{coriolis parameter} \\
g &: \text{acceleration of gravity} \\
A_h &: \text{kinematics mixing coefficient} \\
C &: \text{Chezy coefficient} \\
F_x, F_y &: \text{external forces acting on unit mass} \\
t &: \text{time} \end{align*} \]

### 3 Numerical solution method

According to the calculation scheme shown in Figure 1, if equations (1) and (3) are organised as a function by writing continuity equation at \((i,j)\) points and momentum equation at \((i+1/2, j)\) points for "n" liquid points on \(j\) axis at \(x\)-direction, the non-linear partial differential equation group having "n" unknowns of which the unknowns are velocity components "\(u\)" and water elevations "\(\eta\)".

\[ \begin{align*} f_1(x_1, x_2, x_3, \ldots, x_n) &= 0 \\
f_2(x_1, x_2, x_3, \ldots, x_n) &= 0 \\
&\vdots \\
f_n(x_1, x_2, x_3, \ldots, x_n) &= 0 \\
\end{align*} \]  

or in vector form:

\[ F(x_1, x_2, x_3, \ldots, x_n) = 0 \]  

may be written.

Equation (5) is opened to Taylor series approaching to:

\[ X_o = \begin{bmatrix} x_1^o \\ x_2^o \\ x_3^o \\ \vdots \\ x_n^o \end{bmatrix} \]  

and by taking the first two terms the following equation may be written:

\[ F(X) \equiv F(X^o) + \sum \left( \frac{\partial F}{\partial x_i} \right)_{x_o} (x_i - x_i^o) \]
This equation may be organized in the following form to make consecutive calculation possible.

$$F(X_{k+1}) = F(X_k) + J(X_k) \Delta X$$  \hspace{1cm} (8)$$

In the above equation:

$$\Delta X = X_{k+1} - X_k$$ \hspace{1cm} (8.1)$$

$$J(X_k) = \begin{bmatrix}
\frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\
\frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_n} \\
\cdots & \cdots & \cdots & \cdots \\
\frac{\partial f_n}{\partial x_1} & \frac{\partial f_n}{\partial x_2} & \cdots & \frac{\partial f_n}{\partial x_n}
\end{bmatrix}$$ \hspace{1cm} (8.3)$$

$$J$$ is the Jacobian matrix of function $$f$$ with $$n$$ variables [7]. If $$X_{k+1}$$ vector is the solution of the given equation system $$F(X_{k+1}) = 0$$ is written and this quantity is applied to equation (8):

$$J(X_k) \cdot \Delta X = -F(X_k), \quad k=0,1,2,\ldots$$ \hspace{1cm} (9)$$

may be written. Equation (9) is a linear algebraic equation system of which the unknown vector is $$\Delta X$$. The coefficients matrix of this equation system is tridiagonal and the coefficients matrix is reduced to multipliers as $$A = L U$$ to find the unknown matrix and the computation is repeated until sufficient convergence is reached. This algorithm is called Newton-Raphson method.

When the algorithm described above is repeated for all $$j$$ axes on $$x$$-direction the velocity components "u" and water elevations "$$\eta$$" for all points are obtained. Similarly when the equations (2) and (3) are organized as function by writing continuity equations at $$(i,j)$$ points, momentum equation at $$(i, j+1/2)$$ points for $$n$$ liquid points which are on $$i$$ axis at $$y$$ direction, the non-linear partial differential equations with "$$n$$" unknowns whose unknowns are velocity components "v" and
surface elevations $\eta$ are obtained. So the unknown velocity components and surface elevations have been calculated by using synchronic values.

\begin{figure}
\centering
\includegraphics[width=\textwidth]{grid_system.png}
\caption{Grid system used in the model.}
\end{figure}

4 Application

The example of which the geometry and water depths are given in Figure 2 is solved for various wind directions and speeds beginning with $\Delta t = 60$ seconds making Courant number less than 1 and increasing in the order 120, 240, 480 by choosing twice the former time step. But, significant change in stream formations did not occur. The same procedure is repeated for examples having different geometry and similar results are obtained. In Figures 3, 4 and 5 the current formations obtained at the end of $36^{th}$ hour by choosing $\Delta t = 60, 1800, 3600$ seconds are presented graphically. In figure 6 the required computation times for different Courant numbers are given.
Figure 2: Water depths and geometry for the sample solution

Simulated depth-averaged velocities
\( (\Delta t = 60 \text{ s}; Cr = 0.592) \)

Figure 3: Velocity field for \( \Delta t = 60 \) seconds and Courant number = 0.592
Simulated depth-averaged velocities
(\(\Delta t = 1800\) s; \(C_r = 17.828\))

Figure 4: Velocity field for \(\Delta t = 1800\) seconds and Courant number = 17.828

Simulated depth-averaged velocities
(\(\Delta t = 3600\) s; \(C_r = 35.656\))

Figure 5: Velocity field for \(\Delta t = 3600\) seconds and Courant number = 35.656
Figure 6: Variation of computation time by Courant number.

5 Conclusions

In the solution of the momentum and continuity equations expressing velocity and elevation changes of lake and coastal water with explicit finite differences solution is possible when Courant number is less than 1. But in the implicit models although the solution is stable for all Courant numbers while there isn't significant change in the results for Courant number ≤ 5, the sensibility of the results for bigger Courant numbers decreases and digression from real values occur [5]. In the presented study because an iterative solution algorithm is used by writing the non-linear terms in the equations synchronically, the time step Δt between two solution steps can be chosen large, for the large values of the Courant number like 30 -35 significant difference in the results is not seen. But, as can be seen from Figure 5 a large decrease in computation time is seen. So, long-term wave formations can be obtained, especially the hydrodynamic values needed for monitoring especially seasonal and yearly water quality changes may be found.
References


