Optimal numerical analysis of branched cracks in biaxial stress fields

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Abstract

PWR steam generator tubing made of Inconel 600 is subjected to the stress corrosion cracking, which produces highly irregular shapes with a high degree of crack branching.

The state-of-the-art analytical and experimental efforts in linear elastic fracture mechanics are mostly focussed on simple shaped cracks in uniaxial stress field. While only a few solutions exist for simple shaped and kinked cracks in biaxial stress fields, with solutions of branched cracks currently limited to equibiaxial stress field, numerical analysis using methods such as finite or boundary element method may provide useful and cost effective solutions. However, accurate analysis of branched cracks may require very fine meshes and, consequently, excessively high computational efforts.

The paper discusses some possible time and resources efficient strategies of numerical modeling of branched cracks in general biaxial stress field using the general-purpose finite element code ABAQUS.

The strategies discussed are based on J-integral and stress intensity factor solutions. Different mesh densities are used in the sensitivity analysis to achieve optimal results. The precision of the obtained numerical results is compared to reference solutions from the literature and reasonable agreement is shown.

The main contribution of the paper is an optimal numerical strategy, which maximizes the accuracy of the results at as low computational efforts as feasible. The selected optimal strategy was developed to be used in the future simulations of large networks of inter-granular stress corrosion cracks at the grain-size scale.
1. Mathematical Model

The inter-granular stress corrosion cracking is one of the main causes for early retirement of tubes in steam generators of PWR nuclear power plants [4]. Combined influence of mechanical loads and aggressive environment causes the development of random crack networks. Typical pattern obtained by numerical simulation of inter-granular crack propagation, which is in agreement with metallographical analyses of pulled out tubes from the steam generators [6], is presented in Figure 1.

Figure 1: Simulated crack pattern within a radial slice of a steam generator tube [6]

The most important part of the simulated crack growth (Figure 1) is – besides of realization of the random grain structure – a model of crack initialization and propagation. Adequate crack initialization is obtained by a random process, which considers for example the contact of material with the aggressive medium and orientation of stress field. The crack propagation is also a random process assuming more frequent propagation of cracks with larger stress intensity factors. Large degree of branching and relatively important interactions between neighboring cracks make the use of available methods of linear elastic fracture mechanics [5] hardly possible.

At the moment, the simulations of initialization and propagation of crack patterns are based upon estimation of stress intensity factors by appropriate empirical models, which correlate the actual crack shape with a simple equivalent replacement crack [9]. This approach is – like any interpolation – unpredictable, when unknown crack forms are considered.

Another possibility is direct numerical analysis of stress intensity factors of all the cracks of the crack pattern (Figure 1). The aim of this paper is to optimize the analysis of the stress intensity factors of branched cracks in the biaxial stress field with finite element method [1]. The goal of optimization is to obtain as
accurate stress intensity factors as possible with as low computational effort as feasible. The development of the automatic mesh generation and simulations of the entire pattern of cracks are closely linked future tasks, based on the results discussed in this paper.

2. Mathematical Model

2.1 Crack tip loading

Stress intensity factor $K$ is the most frequently used measure of the crack tip loading. It determines the stress tensor $\sigma_{ij}$ around the crack tip with known geometry [5]:

$$\sigma_{ij} = \frac{K}{\sqrt{2\pi r}} f_{ij}(\vartheta)$$

(1)

where $r$ and $\vartheta$ are polar coordinates of a point of interest ($r = 0$ at the crack tip) and $f_{ij}$ is weight function, which basically depends on the geometry of the crack. According to the type of the crack loading, two independent stress intensity factors $K_1$ (opening) and $K_{II}$ (shear) [5, 3] are used in planar problems.

Stresses or displacements in at least one point should be known to determine stress intensity factors $K_1$ and $K_{II}$. However, the singularity of the stress tensor around crack tip (eqn 1) makes numerical determination of the stress intensity factor with finite element method rather difficult. The estimates of stress intensity factors $K_1$ and $K_{II}$ must be extrapolated towards the crack tip. Therefore models with very fine model meshing are needed to support the extrapolation of appropriate accuracy.

2.2 $J$-integral

Better numerical stability of the results could be obtained using energy quantities, such as J-integral [5, 3]:

$$J = \int_I \left( W d\gamma - T_i \frac{\partial u}{\partial x} ds \right) ,$$

(2)

where $I$ is an arbitrary closed integration contour around crack tip, $T_i$ is stress vector perpendicular to curve $I$, $u$ is displacement within direction $x$ and $W$ is deformation energy.

State-of-the-art techniques for numerical evaluation of J-integral value are based upon integration over area (or volume for three-dimensional cases) instead of integration along the curve (Green’s theorem) [10]. Additional dimension of integration contributes to the stability of results obtained using coarse meshes and therefore reduces computational efforts.
2.3 Decomposition of J-integral to stress intensity factors $K_1$ and $K_{II}$

In elastic fracture mechanics the $J$-integral and stress intensity factors are related as:

$$J = \frac{K^2}{E'} \quad E' = \begin{cases} E & \text{plain strain} \\ \frac{E}{1-\nu^2} & \text{plain stress} \end{cases}$$

$$K^2 = K_1^2 + K_{II}^2. \quad (3)$$

Exact decomposition of $J$-integral into the stress intensity factors $K_1$ and $K_{II}$ is given in the literature [11]. In general, the stress field should be divided into symmetrical and anti-symmetrical part, resulting in symmetrical and anti-symmetrical part of $J$-integral. In our case (Figure 1), such decomposition should be repeated for every crack tip. This would exceed available computational capacity. Therefore, we implemented and tested two decompositions, which are described below.

2.3.1 Decomposition using displacements ahead of crack tip

Finite element method enables calculations of displacements around the crack tip. Displacements $u_x$ and $u_y$ at point $T (r, \theta)$ are given as:

$$u_x = \frac{1 + \nu}{2E} \sqrt{\frac{r}{2\pi}} (f_{xI} \cdot K_I + f_{xII} \cdot K_{II}) \quad u_y = \frac{1 + \nu}{2E} \sqrt{\frac{r}{2\pi}} (f_{yI} \cdot K_I + f_{yII} \cdot K_{II}), \quad (4)$$

where $\nu$ is Poisson’s ratio, $E$ is Young’s module and $f$ is weight factor with index denoting the coordinate ($x$, $y$) and the loading type of the crack (I, II). Stress intensity factors $K_I$ and $K_{II}$ depend upon the displacement field, the size and the geometry of the crack. Values of stress intensity factors $K_I$ and $K_{II}$ can be calculated using:

$$K_I = \frac{f_{yII} \cdot u_x - f_{xII} \cdot u_y}{f_{xI} \cdot f_{yII} - f_{xII} \cdot f_{yI}} \cdot \frac{2E}{1 + \sqrt{\frac{2\pi}{r}}} \quad K_{II} = \frac{f_{xI} \cdot u_y - f_{yI} \cdot u_x}{f_{xI} \cdot f_{yII} - f_{xII} \cdot f_{yI}} \cdot \frac{2E}{1 + \sqrt{\frac{2\pi}{r}}}. \quad (5)$$

Numerical values of stress intensity factors $K_I$ and $K_{II}$ depend strongly on position of the chosen point with respect to the crack tip. Therefore the ratio of both stress intensity factors $k$ was introduced:

$$k = \frac{K_{II}}{K_I} = \frac{f_{xI} \cdot u_y - f_{yI} \cdot u_x}{f_{yII} \cdot u_x - f_{xII} \cdot u_y}. \quad (6)$$

Ratio $k$ was found to be reasonably stable, especially at nodes, which are collinear with observed crack tips. Hence, there is no need for extrapolation of stress intensity factors $K_I$ and $K_{II}$ towards the crack tip. Apart from ratio $k$, the value of $J$-integral, which is a sum of contributions of $K_I$ and $K_{II}$ (eqn 5), is needed for decomposition. Stress intensity factors $K_I$ and $K_{II}$ are then calculated:
where $J_{num}$ represents a numerically obtained value of $J$-integral.

2.3.2 Decomposition using displacements at crack surfaces

Similar calculation with use of ratio $k$ can be used if instead partial calculation of stress intensity factor, calculation displacements at the opposite crack surfaces is used. Ratio $k$ is defined as [2]:

$$ k = \lim_{r \to 0} \frac{\tilde{t} (\tilde{u}^* - \tilde{u}^-)}{\tilde{n} (\tilde{u}^* - \tilde{u}^-)} = \lim_{r \to 0} \frac{\varepsilon_x (u_x^* - u_x^-) + \varepsilon_y (u_y^* - u_y^-)}{-\varepsilon_x (u_y^* - u_y^-) + \varepsilon_y (u_x^* - u_x^-)}, $$

where $r$ is distance from crack tip to the calculation point, $\tilde{t} = (\varepsilon_x, \varepsilon_y)$ is a tangent vector, $\tilde{n} = (-\varepsilon_y, \varepsilon_x)$ is a normal vector at a crack tip, $\tilde{u}^*$ and $\tilde{u}^-$ are displacements at crack surfaces.

Two different decompositions mentioned above are easy to use. Furthermore, they are universal and their use is not limited only to ABAQUS or any other finite element code but it’s possible to use them with other numerical methods.

3. Numerical Example

A numerical example of a branched crack in an equi-biaxial stress field with sensitivity analyses of different mesh densities was performed. Furthermore, both decompositions were compared and accuracy of results was established. Reference results were obtained by calculation with boundary element method [9].

Figure 2: Coarse (left) and fine (right) finite element mesh of a branched crack
The crack was modeled with meshes of different densities. The finest and the coarsest meshes used in calculations are shown in Figure 2. As a measure of the mesh density typical length of element at the crack tip was used. Therefore the finest mesh has a relative size of a typical element equal to 12.5% (8 elements at each of the crack surfaces) and the coarsest mesh has a relative size of a typical element equal to 100% of the crack length (one element at each of the crack surfaces).

The calculation was performed using finite element code ABAQUS/Standard. Meshes around crack tips were modeled using collapsed singular elements. Singular elements accounted for the $1/\sqrt{r}$ singularity of the displacements at the crack tip [1, 8].

Figure 3 presents the relative error of ratio $k$ obtained by decomposition using displacements ahead of crack tip as a function of different relative sizes of typical element. Relative error of 0% was assumed here for reference results [9].

The decomposition is used for three different crack tips in the branched model. Accuracy of the numerical results is within 10% for the finest (12.5%) mesh and within 400% for the coarse (relative size of typical element of 100%) mesh. It is also shown that reasonable accuracy of about 10–15% is obtained with relative size of typical element of 33%.

![Figure 3: Relative error of ratio $k$ obtained by decomposition using displacements ahead of crack tip](image)

Figure 4 presents the relative error of ratio $k$ obtained by decomposition using displacements at crack surfaces as a function of different relative size of typical element. Again, relative error of 0% was assumed here for reference values results. The decomposition using displacements at crack surfaces was applied for three different crack tips of the branched crack.
Accuracy of the numerical results is within 5% for the finest (12.5%) mesh and within 25% for the coarse (relative size of typical element of 100%) mesh. It is also shown that reasonable accuracy of about 10% is obtained with relative size of typical element of 33%.

Figure 4 also shows that decomposition using displacements at crack surfaces is more accurate than decomposition using displacements ahead of crack tip, especially when very coarse meshes (large relative size of typical element) are used. The difference at finer meshes is less pronounced.

![Graph showing relative error of ratio k obtained by decomposition using displacements at crack surfaces](image)

**Figure 4:** Relative error of ratio $k$ obtained by decomposition using displacements at crack surfaces

Similar convergence and accuracy of numerical results was also obtained while analyzing straight and kinked cracks with known closed form solutions [8].

Computational efforts are also important when the optimal numerical analysis in a case of large networks of inter-granular stress corrosion cracks is concerned. The analyses were performed on workstation Sun Sparc Station 20 without multiprocessing. Time needed for finite element (FE) calculation is one of the measures of computational efforts.

Only finite element calculation without postprocessing of data was taken into consideration. Postprocessing is not a significant part of the process time.

Figure 5 presents the values of ratio $k$ obtained by both decompositions as a function of time needed for FE calculation.
Comparison of the results shows fast increase in accuracy with increase of time needed for FE calculation (coarse mesh). If the mesh is refined and time needed for FE calculation is increased (mesh with relative size of typical element over 30%) the accuracy improves (error within 10%). With even finer meshes better results can be obtained, but the time needed for FE calculation increase unproportionally. In case of large networks of cracks as used in future simulations errors of 5 – 10% are acceptable, therefore we recommend the use of meshes with relative size of typical element up to 30%.

4. Summary

The paper discusses optimized analysis of stress intensity factors of branched cracks in biaxial stress field using finite element method. Analyses were made by finite element code ABAQUS and two decomposition of J-integrals. The following actions are recommended for more accurate results:

- Meshes with typical element length not larger than 30% of crack length should be used for reasonably accurate results (within 10%).
- Decomposition using displacements at crack surfaces should be used for decomposition of J-integral with coarser mesh (relative size of typical element 50% or more). Even with one element at each of the crack surfaces (relative size of typical element 100%) the accuracy is within 25%.
- For finer mesh (with relative size of typical element 30% or less), decomposition of J-integral using displacements ahead of crack tip and decomposition using displacements at crack surfaces are equally accurate.
Similar convergence and accuracy of numerical results was also obtained while analyzing straight and kinked cracks with known closed form solutions [8]. Future work will focus on simulations of large networks of inter-granular stress corrosion cracks at the grain-size scale using incomplete random tessellation. The optimal strategy discussed in this paper will be used as a basis.

5. Acknowledgment

Authors wish to thank to Ministry of Science and Technology of Slovenia for financial support and to Dr. Heinz Riesch-Oppermann (Forschungszentrum Karlsruhe) for numerous fruitful discussions.

6. Literature

Boundary Element Technology

