



Computation of optimal Bayesian credible sets for the binomial and Poisson distributions

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Abstract

In the classical framework of statistical inference, the unknown parameter of interest is considered to be a constant, and a classical confidence interval for the parameter is a random interval (L, U) that contains the unknown parameter with a specified high probability. Such a confidence interval is usually derived from the probability distribution of a pivotal quantity. The problem of computing a classical confidence interval becomes somewhat intractable in case a pivotal quantity cannot be found. A Bayesian confidence interval, which is called a Bayesian credible set in Bayesian literature, is an alternative that does not have this drawback. In Bayesian inference, the parameter of interest itself is considered to be a random variable with a prior probability distribution. One reason that the Bayesian confidence interval estimation is not as popular as its classical version is that its computation can become quite complicated. We present an algorithm based on computational geometry for

approximating optimal Bayes credible intervals. A program to compute the optimal Bayes credible interval is developed in Java. We illustrate our procedure using data simulated from the binomial and Poisson models.

1 Introduction

In classical estimation theory, a random sample $\{x_1, x_2, \dots, x_n\}$ is available from a population with (marginal) probability density function (pdf) $f(x, \theta)$. A confidence interval (L, U) for the parameter θ is a random interval that contains θ with a specified high probability $1 - \alpha$. Such a confidence interval is computed by first finding a 'pivotal quantity' T for θ , which is a function of a sample statistics and the parameter θ such that the mathematical expression for its pdf does not involve θ . An interval (A, B) is then found such that $P(A < T < B) = 1 - \alpha$ which is then transformed into the following probability statement:

$$P(L < \theta < U) = 1 - \alpha.$$

The random interval (L, U) is called a $100(1 - \alpha)\%$ confidence interval for θ . There are two problems with this approach: 1) in many situations, it is not possible to find a pivotal quantity, and 2) the interpretation of a classical confidence interval, after it has been computed for a given sample, poses a problem since θ is a constant (see, for example, Daniel, 1995).

In Bayesian estimation, the parameter of interest θ is considered to be a random variable with a prior pdf $g(\theta)$, and the sample $\{x_1, x_2, \dots, x_n\}$ is collected from the conditional distribution of the random variable θ given x , viz. $f(x|\theta)$. The Bayes theorem is then utilized (see, for example, Berger, 1985) to compute the posterior pdf $g(\theta|x_1, \dots, x_n)$, from which a Bayesian *credibility interval* is calculated from the statement:

$$P(L_B < \theta < U_B) = 1 - \alpha$$

The interpretation of a Bayesian confidence region is straightforward and natural (Press, 1989). A Bayesian credible set clearly is not unique, and a reasonable choice is one which has minimum length. The $100(1 - \alpha)\%$ *highest posterior density* (HPD) credible set is defined as the set that contains only those points θ with the largest posterior pdf (Berger, 1985; Gelman *et al*, 1997; Press, 1989). Bayes credible sets are easier to compute than the classical confidence intervals, especially for cases

in which a pivotal quantity cannot be found; they are, however, not as popular as their classical counterparts: they are rarely mentioned in any applied statistics book, and programs are not available in any of the popular software packages such as MINITAB, SPSS, and SAS. The numerical evaluation of the posterior pdf, however, is not always straightforward. Several methods of approximation of the posterior pdf exist in the Bayesian statistical literature (see, for example, Press, 1989): large sample normal approximation, Lindley's approximation, Tierney-Kadane approximation, numerical integration, and Monte Carlo integration. Ananda *et al.* (1993) computed Bayesian confidence intervals for the product of three normal means using Monte Carlo simulation; computation of the classical confidence interval for this problem is quite complicated. Singh and Singh (1993) derived Bayes slippage tests for identifying outliers in an environmental data set. The present paper uses a geometric algorithm to develop a JAVA program for computing HPD credible sets for the binomial and Poisson distributions. It would not be out of context to mention that the exact classical confidence intervals for these two models is not given in most applied statistics books.

2 HPD Credible Set for the Binomial Proportion

Let X be a random variable which, given p , has a binomial pdf with number of trials n and an unknown success probability p :

$$f(x|p) = \binom{n}{x} p^x (1-p)^{n-x}, 0 \leq p \leq 1, x = 0, 1, \dots, n \quad (1)$$

Determination of a prior density is an important element of Bayesian inference. Several methods exist in the literature for selection of a prior pdf (Berger, 1985). In this paper, we will only consider non-informative priors for the unknown parameter p , and present an algorithm based on computational geometry principles for determining the Bayesian HPD credible sets. A computer program in JAVA was written for computing the Bayesian HPD credible set. There are four choices for a noninformative prior (see Berger, 1985):

$$\begin{aligned} g_1(p) &= 1 \\ g_2(p) &= [p(1-p)]^{-1} \\ g_3(p) &= [p(1-p)]^{-1/2} \\ g_4(p) &= p^p (1-p)^{1-p} \\ (0 \leq p \leq 1 \text{ in each case}) \end{aligned}$$

In this paper we will only consider the priors $g_1(p)$ and $g_2(p)$. The first one was used by Bayes in 1763, and the second one is based on a transformation. The posterior pdf's of the binomial proportion p , given the sample x , for these two prior pdf's, are given by:

$$g_1(p|x) = k_2 p^x (1-p)^{n-x}, \quad 0 \leq p \leq 1, \quad (2)$$

and

$$g_2(p|x) = k_2 p^{x-1} (1-p)^{n-x-1}, \quad 0 \leq p \leq 1, \quad (3)$$

where k_1 and k_2 are the normalizing constants. The above posteriors are clearly beta pdfs, and equal-tailed Bayes credible sets can be computed from (2) or (3) using a table of the incomplete beta function. The HPD Bayesian credible sets can be calculated using the following steps (Berger, 1985):

Step 1: Use a numerical search procedure to find all solutions to the nonlinear equation

$$g(p|x) = c$$

and then construct the sets

$$A(c) = \{p: g(p|x) = c\}$$

Step 2: Compute

$$\int_{A(c)} g(p|x) dp$$

Step 3: Find c^* , the solution of the equation

$$\int_{A(c)} g(p|x) dp = 1 - \alpha$$

for c .

$A(c^*)$ obtained in Step 3 is the $100(1-\alpha)\%$ Bayesian HPD credible set for p . A geometric approach for implementing the above steps is described below:

Geometric Algorithm

Most of the pdf's can be modeled by unimodal (mountain) polygons. Details of modeling a pdf by a unimodal polygon are reported in Gewali *et al.* (1999). We represent the input polygon in a doubly-linked list data structure (see, for example, Preparata and Shamos, 1985) to facilitate the traversal of its boundaries. We sweep a horizontal line from top to bottom to capture the optimum credible set of given confidence. A high level sketch of the algorithm follows:

OPTIMUM CREDIBLE SET ALGORITHM

Step 1: Represent the input unimodal polygon in a doubly connected linked list data structure.

Step 2: Partition the polygon into trapezoids by constructing horizontal chords through the vertices of the polygon.

Step 3: Locate the trapezoid corresponding to the optimal credible set by sweeping a horizontal line from top to bottom.

Step 4: Determine the optimal credible set from the trapezoid located in Step 3 by making use of its geometric structure.

The classical exact confidence interval for p (Zar, 1984) is (L, U) where

$$L = \frac{x}{x + (n - x + 1)F_{\alpha(2), \nu_1, \nu_2}}$$

$$U = \frac{(x + 1)F_{\alpha(2), \nu_1', \nu_2'}}{n - x + (x + 1)F_{\alpha(2), \nu_1', \nu_2'}}$$
(4)

where

$$\nu_1 = 2(n - x + 1), \nu_2 = 2x$$

$$\nu_1' = \nu_2 + 2, \nu_2' = \nu_1 - 2$$

$F_{\alpha(2), \nu_1, \nu_2}$ = 2-sided cut-off point for the F-distribution (Zar, 1984)

Examples:

Table 1: Bayesian HPD Credible Sets and Classical Confidence Intervals for the Binomial Proportion

	90% HPD Credible Set	90% Confidence Interval from (4)	95% HPD Credible Set	95% Confidence Interval from (4)
x=6	L1=0.36 U1=0.81 (Figure 1) L2=0.36 U2=0.85 (Figure 2)	L=0.30 U=0.85	L1=0.32 U1=0.84 L2=0.32 U2=0.88	L=0.26 U=0.88
x=4	L1=0.19 U1=0.64 L2=0.16 U2=0.64	L=0.15 U=0.70	L1=0.16 U1=0.68 L2=0.13 U2=0.69	L=0.12 U=0.74

The statistical software package MINITAB was used to generate samples from a binomial distribution with number of trials 10, and success probability of 0.5. Table 1 shows 90% and 95% HPD credible sets for the two noninformative priors $g_1(p)$ and $g_2(p)$, and also the classical confidence intervals. It should be noticed that the true value of p (0.5) falls inside each of the interval given in Table 1, and that the HPD credible set, in each case, is shorter than the corresponding classical confidence interval for p .

3 Computation of Bayesian HPD Credible Sets for the Poisson Mean

The conditional pdf of the observation X , given the Poisson mean λ , is

$$f(x|\lambda) = \frac{e^{-\lambda} \lambda^x}{x!}, x = 0, 1, 2, \dots \quad (5)$$

We will use Jeffreys' non-informative prior (Berger, 1985), given by:

$$g(\lambda) = [I(\lambda)]^{1/2}$$

where
$$I(\lambda) = -E\left(\frac{\partial^2 \ln f(x|\lambda)}{\partial \lambda^2}\right) = \lambda^{-1/2}$$

is the expected Fisher information for the Poisson model (5). The posterior pdf of λ for Jeffreys' non-informative prior is:

$$g^*(\lambda|x) = Ke^{-\lambda} \lambda^{x-1/2}, \lambda > 0 \quad (6)$$

The Bayesian HPD credible set for λ is computed from (6) using our geometric algorithm described in Section 2. The classical confidence interval $((L, U))$ for the Poisson mean λ are given by (Zar, 1984):

$$\begin{aligned} L &= \frac{\chi^2_{1-\alpha/2, \nu_1}}{2} \\ U &= \frac{\chi^2_{\alpha/2, \nu_2}}{2} \end{aligned} \quad (7)$$

$$\begin{aligned} \nu_1 &= 2x \\ \nu_2 &= 2(x+1) \end{aligned}$$

Examples:

Two pseudo-random samples were generated from a Poisson pdf with mean 5. The Bayesian HPD credible sets and corresponding classical confidence intervals are shown in Table 2.

Table 2: Bayesian HPD Credible Sets and Classical Confidence Intervals for the Poisson Mean

Sample	90% HPD Credible Set	90% Confidence Interval from (7)	95% HPD Credible Set	95% Confidence Interval from (7)
x=3	L=0.69 U=6.18 (Figure 3)	L=0.82 U=6.30	L=0.54 U=6.93	L=0.62 U=7.22
x=5	L=1.92 U=8.75 (Figure 4)	L=1.97 U=9.15	L=1.68 U=9.49	L=1.62 U=10.24

It should be noted that the true value 5 is inside each interval computed, and



also that the Bayesian HPD credible set in each case is shorter than the corresponding classical equal tailed confidence interval.

4 Summary

A geometric algorithm has been used to develop a computer program in JAVA for computing the Bayesian HPD credible sets for a single parameter. Bayesian credible sets against non-informative priors have been computed for the binomial proportion and the Poisson mean. We are currently working on implementing this algorithm for computing the Bayesian credible sets for the lognormal mean. We are also trying to modify our algorithm so as to include the case of a multi-modal posterior distribution. These results will be reported in future papers.

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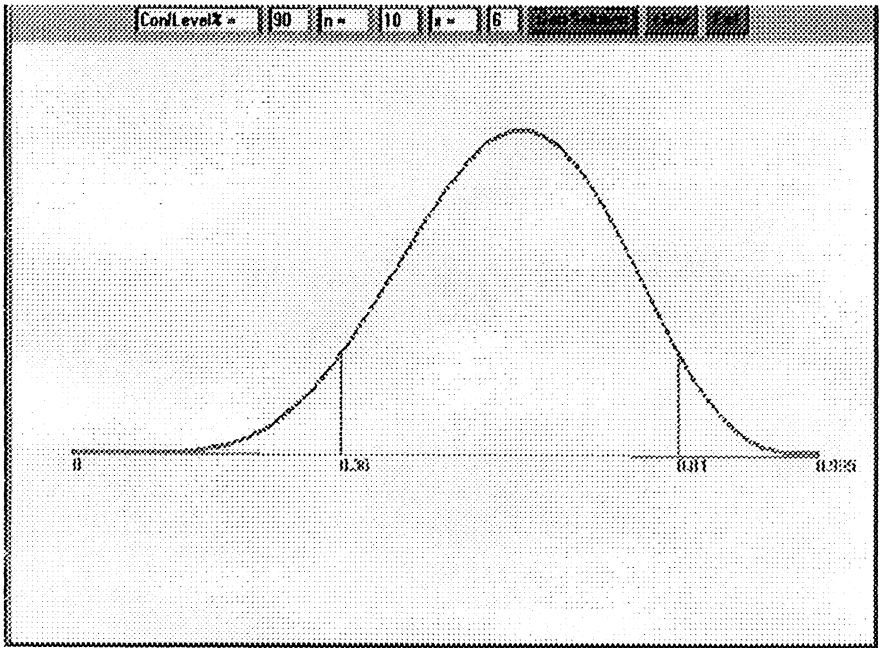


Figure 1: Posterior pdf $g_1^*(p|x=6)$ and 90% HPD Credible Set

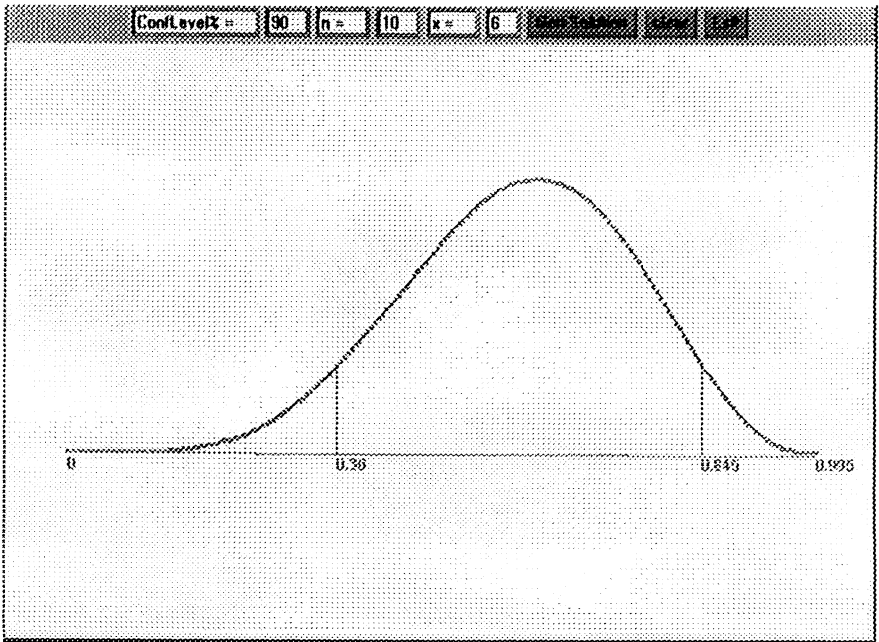


Figure 2: Posterior pdf $g_2^*(p|x=6)$ and 90% HPD Credible Set

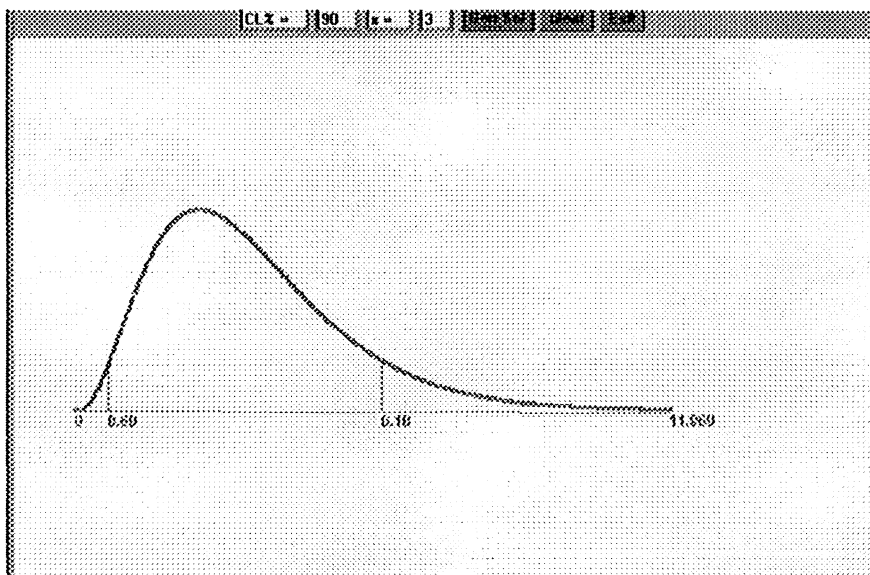


Figure 3: Posterior pdf $g(\lambda|x=3)$ and 90% HPD Credible Set for the Poisson Mean

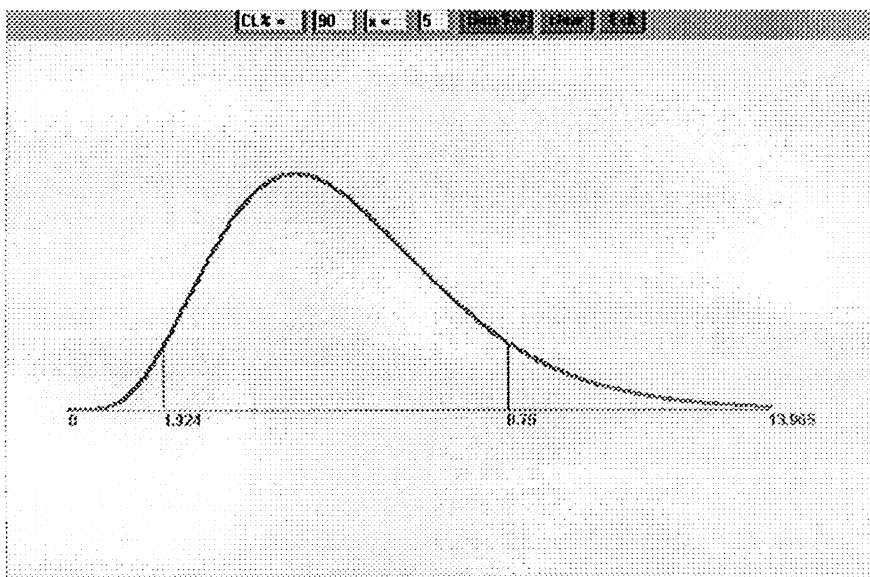


Figure 4: Posterior pdf $g(\lambda|x=5)$ and 90% HPD Credible Set for the Poisson Mean