Decay properties of solutions for some damped wave equations

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Abstract

We study decay properties of global solutions for some non-linear damped wave equations, which include three cases: non-degenerate, degenerate, and inhomogeneous damping. A canonical model for such equations is a generalized Kirchhoff string. We establish decay properties of solutions in energy norm, and give critical damping conditions for a class of non-linear damping functions. In particular we introduce a Lyapunov function which can be used to study the case where damping term strongly depends on time.

1 Introduction

We consider the initial-boundary value problem for the following non-linear wave equation

\[
\begin{aligned}
&u_{tt} - M(\|\nabla u\|^2) \Delta u + Q(t, u_t) + f(u) = 0 \quad \text{in} \ I \times \Omega, \\
&u(0, x) = u_0(x), \quad u_t(0, x) = u_1(x), \quad \text{and} \quad u(t, x) = 0 \quad \text{on} \ \partial \Omega,
\end{aligned}
\]

where \( I = [0, \infty), \) \( \Omega \) is a bounded domain in \( \mathbb{R}^n \) with smooth boundary \( \partial \Omega. \)

When \( M = 1 \) eqn (1) becomes a standard wave equation. The existence and decay properties of wave equations have been discussed in many literatures for a class of non-linear functions \( Q \) and \( f, \) see [1, 2, 4, 5, 6, 9, 10, 11, 12]. On the other hand global non-existence (blow-up) of wave equations has also been studied, see [3]. In particular for the following case

\[
Q = |u_t|^{q-2} u_t \quad \text{and} \quad f = |u|^{p-2} u,
\]

decay properties of global solutions for wave equations is well-known: solutions go to 0 in energy norm exponentially as \( t \to \infty \) for linear damping, and go to 0 in polynomial rate as \( t \to \infty \) for polynomial damping.

The purpose of this paper is to study decay properties of global solutions for the case \( M \neq 1. \) A canonical modal of eqn (1) is:

\[
M(u) = au^\gamma + b, \quad Q(t, u) = (1 + t)^\theta |u|^{q-2} u, \quad \text{and} \quad f(v) = |v|^{p-2} v
\]
with $a, b \geq 0$. If $b = 0$, $a > 0$, (1) is a degenerate wave equation, and if $b > 0$, (1) is a non-degenerate wave equation. For case of $n = 1$ eqn (1) describes the non-linear vibrations of an elastic string, that is,

$$ \rho u_{tt} + \delta u_t = \left\{ p_0 + \frac{Eh}{2L} \int_0^L (u_x)^2 dx \right\} u_{xx} + f. $$

When $\delta = f = 0$, eqn (3) was introduced by Kirchhoff in 1883, and is called the Kirchhoff string.

In [8] the author established decay properties of global solutions for the case $q = 2$ (linear damping) by a modified version of general theory on the energy decay of hyperbolic equations in [7]. Here we use an energy perturbation method to prove decay properties, that is, we define a new energy function (a Lyapunov function) as follows

$$ \Phi = E + \varepsilon \alpha E^m(u, u_t), $$

where $E$ is the energy defined by solutions of eqn (1). By choosing proper numbers $\alpha$ and $m$, the function $\Phi$ provides us with a generic method to handle the cases: degenerate, non-degenerate equations as well as the inhomogeneous term $(1 + t)^b$. We show decay properties of solutions for eqn (1) in energy norm, and give critical damping condition for a class of non-linear functions $Q$ and $f$.

2 Preliminaries

The existence of global solutions of non-linear wave equations has been of considerable interest to mathematicians and physicists, see [4] for classical results. In general there are two ways to prove existence: Galerkin approximation and fixed point theorem. Recently we also noticed some versions of those two methods had been used to prove existence of equations of type (1), see [8].

In this paper we will not focus on global existence. We shall assume that there are global solutions for non-linear functions $Q$ and $f$ in consideration, and leave global existence to separate study. Now let us define global solutions of eqn (1).

Suppose that

$$ (u(0, x), u_t(0, x)) \in H^1_0(\Omega) \times L^2(\Omega). $$

Define the solution space $\mathcal{K}$ by

$$ \mathcal{K} = \{ \xi : I \times \Omega \to \mathbb{R} \mid \xi \in C(I; H^1_0(\Omega)), \quad \xi' \in C(I; L^2(\Omega)) \cap L^q(I; L^q(\Omega)) \}. $$
Here \( \langle \cdot, \cdot \rangle \) is the dual product over \( \Omega \), \( \| \cdot \|_p \) denotes \( L^p(\Omega) \) norm, and all derivatives involved are in the sense of distributions.

**Definition 2.1.** The function \( u \) is said to be a solution of eqn (1) if it satisfies the following conditions:

1. \( u \in \mathcal{K}; \)

2. Energy Identity: \( \langle Q, u_t \rangle \in L^1_{\text{loc}}[0, \infty) \) and

\[
E(t) = \frac{\|u_t\|^2}{2} + \int_0^t \|\nabla u\|^2 \, M(x) \, dx + \int_\Omega F(u) \, dx \bigg|_0^t = -\int_0^t \langle Q, u_t \rangle \quad f = \frac{\partial F}{\partial u};
\]

3. Distribution Identity:

\[
\langle u_t, \phi \rangle \bigg|_0^t + \int_0^t \left\{ -\langle u_t, \phi_t \rangle + \langle M(\|\nabla u\|_2^2) \nabla u, \nabla \phi \rangle + \langle Q, \phi \rangle + \langle f, \phi \rangle \right\} = 0,
\]

for all \( \phi \in \mathcal{K} \).

**Lemma 2.1 (Sobolev).** If \( u \in H^1_0(\Omega) \), then \( u \in L^q(\Omega) \) with \( 1 \leq q \leq n^* \), and the following inequality holds

\[
\|u\|_q \leq C\|\nabla u\|_2
\]

where \( n^* \) is the Sobolev exponent for \( H^1_0 \), that is, \( n^* = \infty \) for \( n = 1 \), \( 2 < n^* < \infty \) for \( n = 2 \), and \( n^* = 2n/(n-2) \) for \( n \geq 3 \).

### 3 Decay properties

In this section we construct Lyapunov functions to prove decay properties. Similar Lyapunov functions have been used in [11, 12]. Here we are going to deal with three cases: non-degenerate, degenerate, and inhomogeneous equations. The basic idea of construction of Lyapunov functions for these three cases was given in (4). But modification for each case is needed to handle its specialty.

**Case 1: non-degenerate equations**

Without loss of generality we assume that the non-linear function \( M \) is:

\[
M(v) = 1 + av^\gamma,
\]
where \( a \geq 0 \) and \( \gamma \geq 0 \). Consider the following initial-boundary value problem

\[
\begin{aligned}
\begin{cases}
  u_{tt} - (1 + a\|\nabla u\|^2)\Delta u + |u|^q u_t + |u|^{p-2} u = 0 & \text{in } I \times \Omega, \\
  u(0, x) = u_0(x), \quad u_t(0, x) = u_1(x), \quad \text{and } u(t, x) = 0 & \text{on } \partial \Omega.
\end{cases}
\end{aligned}
\]  

(5)

Suppose that \( u \) is a global solution of (5), then the energy function \( E \) is the following

\[
E = \frac{\|u_t\|^2}{2} + \frac{\|\nabla u\|^2}{2} + \frac{a\|\nabla u\|^{2(\gamma+1)}}{2(\gamma+1)} + \frac{\|u\|^p}{p}.
\]  

(6)

By the energy identity we have that \( E(t) \leq E(0) \) for \( t \in [0, \infty) \).

In (4) we choose

\[
\alpha = 1, \quad m = (q - 2)/2,
\]  

(7)

and \( \varepsilon \) is a small positive integer which will be determined later. The choice of \( m \) is critical to the proof of decay properties. It is noticed that the choice in (7) is similar to Lyapunov functions which were used in [11, 12] to prove decay properties of global solutions for regular wave equations, i.e., \( a = 0 \).

**Theorem 3.1.** Suppose that \( 2 \leq q \leq n^* \) and \( 0 \leq \gamma \). Then for \( t \in [0, \infty) \)

\[
E(t) \leq c_1 (1 + t)^{-2/(q-2)} \quad \text{if } q > 2
\]

and

\[
E(t) \leq c_2 \exp(-c_3 t) \quad \text{if } q = 2,
\]

where \( c_1, c_2 \) and \( c_3 \) are positive constants which depend on initial data.

**Proof.** It is obvious

\[
\langle u, u_t \rangle \leq \text{Const. } E.
\]

Hence for small \( \varepsilon \) we have

\[
\frac{1}{2} E \leq \Phi \leq 2E \quad \text{for } t \in [0, \infty).
\]  

(8)

Differentiating \( \Phi \) with respect to \( t \) yields

\[
\Phi' = E' + m\varepsilon E^{m-1} E' \langle u, u_t \rangle + \varepsilon E^m \langle u, u_t \rangle'.
\]  

(9)

According to the distribution identity we obtain

\[
\langle u, u_t \rangle' = \|u_t\|^2 - \|\nabla u\|^2 - a\|\nabla u\|^{2\gamma+2} - \langle |u|^{p-2} u, u \rangle - \langle |u|^{q-2} u_t, u \rangle.
\]
Applying Schwarz’s inequality gives
\[ |\langle u, u_t \rangle| \leq E(0)/2. \]

If \( \varepsilon \) is small enough, then there is a positive constant \( C_1 \) such that
\[ 1 + m\varepsilon E^{m-1}\langle u, u_t \rangle \geq C_1. \]

Thus
\[ \Phi' \leq -C_1||u_t||_q^2 + \varepsilon E^m||u_t||_2^2 - \varepsilon E^m||\nabla u||_2^2 - \varepsilon a E^m||\nabla u||_2^{2(\gamma+1)} - \varepsilon E^m||u||_p^p - \varepsilon E^m\langle u, |u_t|^{q-2}u_t \rangle. \]

By Hölder’s inequality and Young’s inequality
\[ |\langle u, |u_t|^{q-2}u_t \rangle| = \langle (C_2)^{1/q}|u|, (C_2)^{-1/q}|u_t|^{q-1} \rangle \]
\[ \leq C_2||u||_q^q + C_2^{1/(q-1)}||u_t||_q^q, \]

where \( C_2 \) is a positive number. Then we get
\[ \Phi' \leq (-C_1 + C_2^{-1/(q-1)}\varepsilon E^m)||u_t||_q^q + \varepsilon E^m||u_t||_2^2 \]
\[ - \varepsilon E^m(||\nabla u||_2^2 + a||\nabla u||_2^{2(\gamma+1)} - C_2||u||_q^q + ||u||_p^p). \]

By Sobolev’s inequality for small \( C_2 \) we get
\[ ||\nabla u||_2^2 + a||\nabla u||_2^{2(\gamma+1)} - C_2||u||_q^q + ||u||_p^p \geq C_3 E \]
with \( 0 < C_3 < 1 \). Therefore,
\[ \Phi' \leq (-C_1 + C_2^{-1/(q-1)}\varepsilon E^m)||u_t||_q^q + 2\varepsilon E^m||u_t||_2^2 - \varepsilon C_3 E^{m+1} \]

Again applying Hölder’s inequality and Young’s inequality yields
\[ 2\varepsilon E^m||u_t||_2^2 \leq C_4\varepsilon E^{q/2} + C_5\varepsilon||u_t||_q^q \]
where \( C_4 \) is a small positive number. Finally we obtain
\[ \Phi' \leq (-C_1 + C_2^{-1/(q-1)}\varepsilon E^m + C_5\varepsilon)||u_t||_q^q - C_6\varepsilon E^{q/2}. \]

If \( \varepsilon \) is small, then by the above inequality and (8) we get
\[ \Phi' \leq -C_7\varepsilon \Phi^{q/2}. \]

Integrate (12) with respect to \( t \) to complete the proof.
Case 2: degenerate equations

Next we consider the case for which

\[ M(v) = v^\gamma \]

with \( \gamma > 0 \). The initial-boundary value problem (1) now is

\[
\begin{align*}
  \begin{cases}
    u_{tt} - \| \nabla u \|_2^{2\gamma} \Delta u + |u_t|^{q-2} u_t + |u|^{p-2} u = 0 & \text{in } I \times \Omega, \\
    u(0, x) = u_0(x), \quad u_t(0, x) = u_1(x), \quad \text{and } u(t, x) = 0 & \text{on } \partial \Omega.
  \end{cases}
\end{align*}
\]

(13)

Suppose that \( u \) is a global solution of (13), then the energy function is the following

\[ E = \frac{\| u_t \|_2^2}{2} + \frac{\| \nabla u \|_2^{2(2\gamma+1)}}{2(2\gamma+1)} + \frac{\| u \|_p^p}{p}. \]

(14)

The Lyapunov function defined by (7) can not be applied to (13) because of the degeneracy. We need choose a new \( m \) to deal with the degenerate term \( \| \nabla u \|_2^{2\gamma} \). Hence \( \gamma/(\gamma+1) \) is added to \( m \). Set

\[ \alpha = 1, \quad m = \gamma/(\gamma+1) + (q-2)/2, \]

(15)

and \( \varepsilon \) is a small positive number which will be determined later.

**Theorem 3.2.** Suppose that \( 2 < q < n^* \) and \( 0 < \gamma \leq (q-2)/2 \). Then for \( t \in [0, \infty) \)

\[ E(t) \leq c_1 (1 + t)^{-1/m} \]

where \( c_1 \) is a positive constant which depends on initial data, and \( m \) is defined in (15).

**Proof.** By Schwarz’s inequality we have

\[ \langle u, u_t \rangle \leq \| u \|_2^2 + \| u_t \|_2^2 \leq \text{Const.} \ (E + E^{1/(\gamma+1)}). \]

Hence for small \( \varepsilon \)

\[ \frac{1}{2} E \leq \Phi \leq 2E \quad \text{for} \quad t \in [0, \infty). \]

(16)

According to the proof of Theorem 3.1 we get

\[
\Phi' \leq ( - C_1 + C_2^{-1/(q-1)} \varepsilon E^m ) \| u_t \|_q^q + \varepsilon E^m \| u_t \|_2^2 \\
- \varepsilon E^m \left( \| \nabla u \|_2^{2(\gamma+1)} - C_2 \| u \|_q^q + \| u \|_p^p \right)
\]
where $C_1$ and $C_2$ are positive constants. By Sobolev’s inequality and the assumption $\gamma \leq (q - 2)/2$ we get for small $\varepsilon$

$$\|\nabla u\|_2^{2(\gamma + 1)} - C_2\|u\|_q^2 + \|u\|_p^p \geq C_3 \left(\|\nabla u\|_2^{2(\gamma + 1)/(2\gamma + 2)} + \|u\|_p^p/p\right)$$

with $0 < C_3 < 1$. Therefore,

$$\Phi' \leq \left( - C_1 + C_2^{-1/(q - 1)} \varepsilon E^m \right) \|u_t\|_q^q + 2\varepsilon E^m \|u_t\|_2^2 - \varepsilon C_3 E^{m+1}$$

Applying Hölder’s inequality and Young’s inequality yields

$$2\varepsilon E^{(q-2)/2} \|u_t\|_2^2 \leq C_4 \varepsilon E^{q/2} + C_5 \varepsilon \|u_t\|_q^q$$

where $C_4$ is a small positive number. Finally we obtain

$$\Phi' \leq \left( - C_1 + C_2^{-1/(q - 1)} \varepsilon E^m + C_5 \varepsilon E^{\gamma/(\gamma + 1)} \right) \|u_t\|_q^q - C_6 \varepsilon E^{m+1}.$$ 

If $\varepsilon$ is small, then by the above inequality and (16) we get

$$\Phi' \leq -C_7 \varepsilon \Phi^{m+1}.$$ 

Integrate (18) to complete the proof.

**Remark 3.1.** The result of Theorem 3.1 does not include the following case

$$M(x) = x^\gamma \text{ with } \gamma > 0, \text{ and } Q = u_t.$$ 

Without further constraint on $f$ the above proof will fail. More precisely the estimate in (17) will not hold. But if we add a linear term to $f$ then the result of Theorem 3.2 still holds. In particular we claim that Theorem 3.2 holds on global solutions for eqn (13) when $f = u + |u|^{p-2}u$.

**Remark 3.2.** When $q > n^*$ ($n \geq 3$), by the main theorem of [9] we have $E \to 0$ as $t \to \infty$ if (1) is a standard wave equation, and $p \geq q$. But no decay property of solutions could be derived from that theorem. Here we claim that if $p \geq q$, $pq - 3p + 2 \geq 0$, and $q > n^*$, then Theorem 3.1 and Theorem 3.2 still hold.

**Case 3: inhomogeneous equations**

Finally we study the case where the damping term strongly depends on time and the equation is non-degenerate. The typical damping is of the form

$$Q = (1 + t)^\theta |u_t|^{q-2}u_t.$$
Consider the following initial-boundary value problem

\[
\begin{cases}
    u_{tt} - (1 + a\|\nabla u\|_2^{2\gamma}) \Delta u + \delta(t) |u_t|^{q-2} u_t + |u|^{p-2} u = 0 \quad \text{in } I \times \Omega, \\
    u(0, x) = u_0(x), \quad u_t(0, x) = u_1(x), \quad \text{and } u(t, x) = 0 \quad \text{on } \partial \Omega.
\end{cases}
\]  

(19)

In order to handle the inhomogeneous term \(\delta\) we choose \(m = (q - 2)/2\), and \(\alpha\) is an absolutely continuous function of time satisfying

\[|\alpha'| \leq K\alpha.\]

Again \(\varepsilon\) is small and will be determined later.

**Theorem 3.3.** Suppose that \(2 \leq q \leq n^*\) and \(0 \leq \gamma\) and in addition \(\alpha, \delta\) satisfy

\[\text{Const. } \alpha \leq \delta \quad \text{and} \quad \alpha \delta^{1/(q-1)} \leq \text{Const.}.\]  

(20)

Then for \(t \in [0, \infty)\)

\[E(t) \leq c_1 \left(1 + \int_0^t \alpha \right)^{-2/(q-2)} \quad \text{if } q > 2\]

and

\[E(t) \leq c_2 \exp \left(-c_3 \int_0^t \alpha \right) \quad \text{if } q = 2,
\]

where \(c_1, c_2\) and \(c_3\) are positive constants which depend on initial data.

**Proof.** Differentiating \(\Phi\) with respect to \(t\) yields

\[\Phi' = E' + m\varepsilon\alpha E^{m-1} E'\langle u, u_t \rangle + \varepsilon \alpha' E^m \langle u, u_t \rangle + \varepsilon \alpha E^m \langle u, u_t \rangle'.\]

(21)

According to (20) the function \(\alpha\) is bounded. Then

\[\Phi' \leq -C_1 \delta \|u_t\|_2^2 + \varepsilon \alpha E^m \|u_t\|_2^2 - \varepsilon \alpha E^m \|\nabla u\|_2^2 - \varepsilon \alpha E^m \|\nabla u\|_2^{2(\gamma+1)} - \varepsilon \alpha E^m \|u\|_p^p - \varepsilon \alpha \delta E^m \langle u, |u_t|^{q-2} u_t \rangle + \varepsilon \alpha' E^m \langle u, u_t \rangle.
\]

(22)

By Hölder’s and Young’s inequalities, and the assumption \(|\alpha'| \leq K\alpha\) we get

\[|\alpha' \langle u, u_t \rangle| \leq \alpha \omega K \|u\|_2^2 + \alpha \omega^{-1} K \|u_t\|_2^2.
\]

Also instead of (11) we shall use the following estimate

\[\delta \langle u, |u_t|^{q-2} u_t \rangle = \delta \langle (C_2/\delta)^{1/q} |u_t|, (C_2/\delta)^{-1/q} |u_t|^{q-1} \rangle \leq C_2 \|u\|_q^q + C_2^{-1/(q-1)} \delta^{q/(q-1)} \|u_t\|_q^q,
\]

(23)
Thus we have
\[
\Phi' \leq \left( -C_1 \delta + C_2^{-1/(q-1)} \varepsilon \alpha \delta^{q/(q-1)} E^m \right) \|u_t\|_q^q + \varepsilon \alpha (1 + \omega^{-1} K) E^m \|u_t\|_2^2
\]
\[- \varepsilon E^m \left( \|\nabla u\|_2^2 + a \|\nabla u\|_2^{2(\gamma+1)} - \omega K \|u\|_2^2 - C_2 \|u\|_q^q + \|u\|_p^p \right) .
\]
For small \( \omega \) and \( C_2 \) we obtain
\[
\Phi' \leq \left( -C_1 \delta + C_2^{-1/(q-1)} \varepsilon \alpha \delta^{q/(q-1)} E^m \right) \|u_t\|_q^q + \varepsilon \alpha C_4 E^m \|u_t\|_2^2 - \varepsilon C_3 E^{m+1}
\]
with \( 0 < C_3 < 1 \) and \( C_4 = 2 + \omega^{-1} K \). Applying Hölder’s inequality and Young’s inequality yields
\[
C_4 E^m \|u_t\|_2^2 \leq C_5 E^{q/2} + C_6 \|u_t\|_q^q
\]
where \( C_5 \) is a small positive number. Finally we obtain
\[
\Phi' \leq \left( -C_1 \delta + C_2^{-1/(q-1)} \varepsilon \alpha \delta^{q/(q-1)} E^m + C_6 \varepsilon \alpha \right) \|u_t\|_q^q - C_7 \varepsilon \alpha E^{q/2}.
\]
If \( \varepsilon \) is small, then
\[
\Phi' \leq -C_8 \varepsilon \alpha \Phi^{m+1}.
\]
Therefore the proof is completed.

**Remark 3.3.** To apply the above approach to degenerate inhomogeneous equations has essential difficulty. If \( f = u + |u|^{p-2} u \) in (19) then the conclusion of Theorem 3.3 still holds. But for \( f = |u|^{p-2} u \) we shall leave it to further study.

At last we shall look at a concrete case. Suppose that
\[
Q = (1 + t)^\theta |u_t|^{q-2} u_t, \quad f = |u|^{p-2} u,
\]
where \( q, \gamma \) satisfy the assumption in Theorem 3.3. We choose
\[
\delta = (1 + t)^\theta
\]
and
\[
\alpha = \frac{1}{(1 + t)^{-\theta} + (1 + t)^{\theta/(q-1)}}.
\]
It is easy to see that \( \alpha \) verifies (20). Therefore, \( E \to 0 \) when \(-1 \leq \theta \leq q-1\), and the decay rate of \( E \) is given by Theorem 3.3. On the other hand, for the case \( q = 2 \), it is known that if \( \theta < -1 \) then \( E \not\to 0 \). There is also an example for the case \( (q = 2) \) in section 5 of [9]. It shows that if \( \theta > 1 \), then \( E \) is not necessary to go to zero.
References


