Boundary element analysis of the productivity of complex petroleum well configurations

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Abstract

This paper presents a Boundary Element approach for predicting the productivity of oil wells located arbitrarily in complex configurations within irregularly shaped reservoirs. The integral equations are written for boundary points as well as for the locations of the wells which are treated as point sources and sinks with specified pressures but unknown strengths. Using this approach, the solution to the resulting matrix gives the values of the nodal boundary pressures and their normal derivatives, as well as the unknown flow rates of all the wells. Example BEM solutions are verified against known analytical solutions. Applications to regular four-spot, five-spot, seven-spot and nine-spot drilling patterns in order to determine which pattern produces the most oil for a given field are presented. Other potential uses include (i) the calculation of the production of individual wells within leases in a multiple lease reservoir and (ii) the identification of candidate wells in a field that may need work-over by comparing the predicted production rates with the actual field production rates.

1 Introduction

The productivity (flow rate) of a well in the presence of other wells in a bounded oil field is a function of the pressure at its well-bore, the bottom-hole pressures or flow rates of the other wells, and the prevailing reservoir and boundary pressures. The unknown reservoir pressure depends on the reservoir shape, boundary conditions, the number of wells (sources and/or sinks) and their operating well-bore pressures. Analytical expressions for the productivities of wells or groups of wells are restricted to radial and rectangular reservoirs. Muskat\(^1\) gives analytical expressions for the productivity of several well clusters in circular
reservoirs. He uses an average external pressure instead of the actual external pressures in calculating the flow rates of the wells. Furthermore, Muskat used the same well-bore pressure for every well.

The productivity of wells located in irregularly shaped reservoir can be obtained by methods of numerical reservoir simulation. Conventional Petroleum Simulators require the domain of the problem to be divided into a number of finite sub-domains. Furthermore, detailed properties of the domain (reservoir) are needed as data to run the simulators. Thus, commercial petroleum simulators can be expensive and involve a large amount of data input.

There is a need for a simple but elegant and fast method to calculate complex well productivities without resorting to finite difference simulators. This study presents such an approach. It is based on the Boundary Element Method (BEM) but with the modification that the integral equations are written for boundary points as well as for the locations of the point sources and/or sinks (producer and/or injector wells). A solution of the resulting equation gives the value of flow rate $q$ for each of the well directly as well as the unknown boundary values of $p$ and $dp/dn$.

**2 Problem Formulation**

Consider a hypothetical two-dimensional homogeneous reservoir having $NSS$ sources and/or sinks located randomly within an arbitrarily shaped reservoir. Assume that the reservoir is undergoing steady-state flow with reservoir pressure above the bubble point i.e. undersaturated condition. Assume that a single phase fluid (oil) having small (and constant) compressibility and constant viscosity is flowing in the system. Third, the reservoir has a uniform thickness and has a finite boundary and finally, gravitational effects are negligible.

The differential equation describing the unknown functions i.e. pressure, at all points in the reservoir is obtained by the introduction of Darcy’s law into the continuity equation. By imposing the conditions and assumptions discussed above, the differential equation describing the pressure distribution in the reservoir is:

$$\frac{\partial^2 p}{\partial X^2} + \frac{\partial^2 p}{\partial Y^2} + \frac{\mu}{k} \sum_{m=1}^{NSS} q_m \delta(X-X_m,Y-Y_m) = 0$$

where $p$ is pressure, $\mu$ is the dynamic viscosity of the fluid, $k$ is the permeability, $q_m$ is the flow rate of the $m^{th}$ well per unit area (positive for injectors and negative for producers), $\delta$ is the Dirac delta function, $X, Y$ are coordinates axes, and $X_m,Y_m$ are coordinates of the $m^{th}$ source and/or sink where $m$ goes from 1 to $NSS$.

Eqn (1) can be transformed into an integral equation by multiplying it with the free-space Green’s function and integrating it twice by parts. The free-space Green’s function is also called the fundamental solution and is given as:
where \( r \) is the distance between a field point \((X, Y)\) and a point of application of a unit charge \((X_0, Y_0)\). After standard manipulation, eqn (1) then becomes:

\[
\alpha p (X, Y) = \frac{1}{2\pi} \sum_{j=1}^{N} \frac{\partial p}{\partial n_j} \int_{s_j} \ln\left(\frac{1}{r_{pj}}\right) ds
\]

\[
\frac{1}{2\pi} \sum_{j=1}^{N} P_j \int_{s_j} \frac{\partial}{\partial n} \ln\left(\frac{1}{r_{pj}}\right) ds + \frac{1}{2\pi} \frac{\mu}{k} \sum_{m=1}^{NSS} q_m \ln\left(\frac{1}{r_{pm}}\right)
\]

where the boundary of the reservoir is divided into \( N \) constant elements with constant properties as shown in Figure 1. \( \alpha \) is the included angle at the \( i^{th} \) pivot point. It is assigned a value of \( \frac{1}{4} \) when the pivot point is on a smooth boundary (i.e. not on a corner), and a value of 1 when the pivot point is inside the problem domain. For simplicity, let

\[
G_{ij} = \frac{1}{2\pi} \int_{s_j} \ln\left(\frac{1}{r_{ij}}\right) ds
\]

\[
H_{ij} = \frac{1}{2\pi} \int_{s_j} \frac{\partial}{\partial n} \ln\left(\frac{1}{r_{ij}}\right) ds
\]

\[
GSS_{pm} = \frac{1}{2\pi} \ln\left(\frac{1}{r_{pm}}\right)
\]
where \( X_i, Y_i \) are coordinates of any pivot point, \( r_{ij} \) is the distance between the pivot point and the \( j^{th} \) element where \( j \) runs from 1 to \( N \), and \( r_{im} \) is the distance between the pivot point and the \( m^{th} \) source and/or sink. Eqn (3) now simplifies to:

\[
\alpha p (X_i, Y_i) = \sum_{j=1}^{N} \frac{\partial p}{\partial n_j} G_{ij} - \sum_{j=1}^{N} p_j H_{ij} + \sum_{m=1}^{NS} q_m GSS_{im} \tag{7}
\]

The boundary of the reservoir, \( S \), can be of the type \( S^p \) or \( S_{dp/dn} \) or a combination of the two types. Over the \( S^p \) type boundary, the pressure \( p \) is specified as constant throughout the element while \( dp/dn \) is unknown. Over the \( S_{dp/dn} \) type boundary, the \( dp/dn \) is prescribed as constant and the pressure \( p \) is unknown. Similarly, the sources and/or sinks can also have known and unknown rates.

Of the \( N \) elements of the boundary of the reservoir, \( N1 \) elements have \( S^p \) type boundary while the remaining \( N2 \) elements have \( S_{dp/dn} \) type boundary. Of the \( NSS \) sources and/or sinks, \( NPSS \) of these have unknown flow rates while the remaining \( NQSS \) have known or specified flow rates. Assuming that the boundary is smooth and by taking the point of application of the unit charge to the boundary (\( \alpha = 1/2 \)), the integral equation describing the pressure at any boundary point \( (X_b, Y_b) \) is now given by:

\[
\frac{1}{2} p(X_b, Y_b) = \sum_{j=1}^{N1} \frac{\partial p}{\partial n_j} G_{bj} + \sum_{j=N1+1}^{N} \frac{\partial \hat{p}}{\partial n_j} G_{bj} - \sum_{j=1}^{N1} \hat{p}_j H_{bj} - \\
\sum_{j=N1+1}^{N} p_j H_{bj} + \sum_{m=1}^{NPSS} q_m GSS_{bm} + \sum_{m=NPSS+1}^{NSS} \hat{q}_m GSS_{bm} \tag{8}
\]

where \( b \) runs from 1 to \( N \) and \( \hat{p} \), \( dp/dn \) and \( \hat{q} \) are the prescribed values for \( p \), \( dp/dn \) and \( q \) (flow rate) respectively. \( G_{bj} \) and \( H_{bj} \) are the expressions given by eqns (4) and (5), except that \( r_{ij} \) is replaced by \( r_{bj} \), which is the distance between boundary pivot node \( b \) and \( j^{th} \) element. \( GSS_{bm} \) is similar to the expression defined in eqn (6) with \( r_{im} \) replaced by \( r_{bm} \), which is the distance between boundary pivot node \( b \) and the \( m^{th} \) source and/or sink. Both the terms \( G_{bj} \) and \( H_{bj} \) are evaluated for all the elements on the boundary, for which two types of integrals (singular and non-singular) need to be considered. The evaluation of both types of integral are treated by Numbere\(^{2,3} \). The sources and/or sinks in the reservoir are handled by treating them as point sources and/or sinks. This way, the \( GSS_{bm} \) terms are calculated analytically without integration.

Apply eqn (8) at all the boundary nodes and move all the terms on the right-hand side to the left-hand side. The system can be written in matrix form as:

\[
[H] [p] - [G] [dp/dn] - [GSS] [q] = 0 \tag{9}
\]
where [H], [G] and [GSS] consist of the coefficients of $H_{bj}, G_{bj}$, and $GSS_{bjm}$ respectively and $[\bar{p}], [\bar{dp}/dn]$ and $[\bar{q}]$ consist of both known and unknown values of $p$, $dp/dn$ and $q$ respectively. A pattern can be seen here where the terms that can combine with $1/2p$ are arranged diagonally over the element of $[H]$.

In this study, the unknown boundary values as well as interior point values can be determined simultaneously. A system of $N + NPSS$ equations with $N + NPSS$ unknowns can be obtained by applying eqn (7) with $\alpha = 1$ at the locations of the sources and/or sinks with the unknown strengths. In this case, the pressure term on the left-hand side of eqn (7) is the well-bore pressure. When eqn (7) is repeated for all $NPSS$ wells with unknown flow rates, however, an equation with $2 \times NPSS$ unknowns results due to the pressures and the flow rates. These $2 \times NPSS$ unknowns can be reduced to $NPSS$ unknowns by assuming that at every well with an unknown flow rate, the well-bore pressure is given. When the pivot point is at a source or sink location a mathematical singularity will occur when integrating over itself. In order to avoid this problem, the minimum distance is set equal to well-bore radius when calculating the well-bore pressure at a well of given rate. This is in line with natural wells which have finite radii. Finally, rearrange all $N$ applications of eqn (8) and $NPSS$ applications of eqn (7) (with $\alpha = 1$) by placing all the known terms to the right-hand side and leaving all the unknown terms at the left hand side. The $N+NPSS$ by $N+NPSS$ system of equations and unknowns are simplified to matrix form and denoted as:

$$[HGGSS] \bar{U} = \bar{A}$$

(10)

$[HGGSS]$ consists of the coefficients $H_{bj}, G_{bj}$ and $GSS_{bjm}$ (from eqn (8)) and $H_{mj}, G_{mj}$ and $GSS_{mjm}$ (from eqn (7)) corresponding to the unknowns to be found where $m'$ goes from 1 to $NSS$. The vector $\bar{U}$ contains all the $N+NPSS$ unknowns of $p$, $dp/dn$ and $q$ and $\bar{A}$ is a vector containing all the known values.

### 3 Validation

This study can handle three types of cases which are:

**Case I:** The flow rates of the wells are specified and independent of their absolute positions. The values of $p$ or $dp/dn$ are calculated. (Matches Muskat’s conditions).

**Case II:** The well-bore pressures are specified. The calculated flow rates are dependent on their absolute positions and the values of $p$ or $dp/dn$, and

**Case III:** Both types of the cases above are present. That is, some wells have specified flow rates while others have specified well-bore pressures.

Case I is ideal for comparison with Muskat’s analytical solutions. However, since the focus of this study was to determine the flow rates of the wells Case II was used to compare with Muskat’s solutions provided that
the well or cluster of wells is placed at the center of the reservoir. This ensures that the pressure is evenly distributed around the boundary of the reservoir. In other words, the pressure is the same at every point on the reservoir boundary. With this assumption, we are now in a position to validate the BEM solutions using Muskat’s analytical solutions for comparison.

Assume a hypothetical circular, homogeneous, undersaturated and steady-state reservoir containing a cluster of \( n \) wells. The reservoir has the following properties:

- \( R = 10,000 \) feet (reservoir radius), \( r_w = 0.25 \) feet (well-bore radius),
- \( k = 100 \) md (absolute permeability), \( \phi = 0.15 \) (reservoir porosity),
- \( h = 35 \) feet (reservoir thickness), \( p_e = 2000 \) psi (external reservoir pressure),
- \( \rho = 62.4 \) lb/ft\(^3\) (reservoir fluid density), \( p_w = 100 \) psi (well-bore pressure),
- \( \mu = 1.0 \) cp (fluid viscosity), and \( d_2 = d_3 = d_4 = 250 \) feet (\( d_n \) is mutual separations between the \( n \) wells in the reservoir)

The center of the cluster is at the reservoir center and the cluster contains two, three or four wells. Muskat suggested that for a single well in the center of a circular reservoir, the flow rate of the well is given by the radial form of Darcy’s law:

\[
q = 7.08 \times 10^{-3} \frac{k h (\hat{p}_e - p_w)}{\mu \ln \left( \frac{R}{r_w} \right)} \tag{11}
\]

where \( \hat{p}_e \) is the average external boundary pressure. For two wells equidistant from the center, the flow rates of the wells are given as:

\[
q_A = q_B = 7.08 \times 10^{-3} \frac{k h (\hat{p}_e - p_w)}{\mu \ln \left( \frac{R^2}{r_w d_2} \right)} \tag{12}
\]

The flow rates for three wells in a centered triangle are given as:

\[
q_A = q_B = q_C = 7.08 \times 10^{-3} \frac{k h (\hat{p}_e - p_w)}{\mu \ln \left( \frac{R^3}{r_w d_3^2} \right)} \tag{13}
\]

For a group of four wells forming a square pattern the flow rate of each well is given as:

\[
q_A = q_B = q_C = q_D = 7.08 \times 10^{-3} \frac{k h (\hat{p}_e - p_w)}{\mu \ln \left( \frac{R^4}{\sqrt{2} r_w d_4^3} \right)} \tag{14}
\]

The flow rates as calculated from Muskat’s analytical equations are compared
with the BEM solution using constant elements and the percentage errors tabulated in Table I. The negligible percentage error in Table I clearly shows that the BEM solutions agree with Muskat’s analytical solutions.

Even though regular well patterns and boundary geometries are presented in these example applications, this was done simply to allow comparison with published analytical solutions. The method is equally applicable to non-pattern well clusters arbitrarily located in reservoirs with irregular boundary shapes.

Table I. Comparison between Muskat’s analytical solution and the BEM solution for circular reservoir

<table>
<thead>
<tr>
<th>Number of Wells</th>
<th>Flow Rates $q$ (bbl/d)</th>
<th>Percentage Error (%)</th>
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<tr>
<td></td>
<td>Muskat ($m$)</td>
<td>BEM ($b$)</td>
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<tr>
<td>1</td>
<td>4443.11</td>
<td>4445.67</td>
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<tr>
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<td>4</td>
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4 Applications

Having developed the tool for calculating well flow rates, it was used to study the effect of changing the well spacing within various drilling patterns and observing the effect on the production rates of the wells in the pattern. This has direct application to the study of in-fill drilling. Consider a square, homogeneous, undersaturated and steady-state reservoir having the following properties:

$L = 4,000$ feet (reservoir length and width), $\mu = 1.0$ cp, $k = 100$ md, $\phi = 0.15$, $p_e = 2000$ psi, $h = 35$ feet, $\rho = 62.4$ lb/ft$^3$, $p_w = 100$ psi and $r_w = 0.25$ feet

In each case, the reservoir is developed first with a single pattern covering the entire area. Then more wells are added to give an increasing number of patterns by reducing the pattern area or well spacing. The four drilling patterns chosen are regular four-spot, five-spot, seven-spot and nine-spot. All wells in this analysis are producers. Figures 2, 3, 4 and 5 show the well arrangements for regular the fully developed four-spot, five-spot, seven-spot and nine-spot drilling patterns respectively. $A$ is the center point of the reservoir while $B$, $C$, $D$, $E$, $F$ and $G$ refer to the first, second, third, fourth, fifth and sixth stages of well development respectively. Except for the seven-spot pattern (Figure 4), $A$ is also represent a well. Notice that Figure 3 (five-spot) has exactly the same number of wells and well locations with Figure 5 (nine-spot). Therefore, we might
expect similar results for these two patterns.

A plot of the total production rate from the reservoir $Q_n$ divided by the flow rate of the single well reservoir case $Q_i$ (using eqn (12)) as a function of number of wells for all four drilling patterns is shown in Figure 6. The curve for a nine-spot pattern falls exactly on the five-spot curve as expected. The slope of all curves decrease as the number of wells increase and eventually approach a straight line as a consequence of the steady-state assumption. A plot of average production per well $Q_{avg}$ as a function of number of wells is shown in Figure 7. As expected, the average production per well drops as the number of wells increases (inversely proportional to the well
Figure 6. $Q_n/Q_1$ as a function of number of wells present in the reservoir

However, the five-spot and the nine-spot have the highest $Q_{avg}$ value for any number of wells present in the reservoir, followed by the four-spot and the seven-spot patterns. This is consistent with Figure 6.

For oilfield applications, the calculated production rates of individual wells can be compared with actual field production rates to determine which wells are performing below expectation. These wells then become candidates for work-over operations. Due to the ease of adapting the BEM procedure to zoned porous media,
this technique can be used to calculate the production rates of wells within individual leases in multiple lease reservoirs.

5 Conclusions

The concept of formulating the Boundary Element Equations at source and/or sink points as well as at boundary node points was investigated and found to give excellent results. The formulation has the advantage of calculating the unknown source and/or sink rates directly as part of the matrix solution. The negligible percentage error in Table I indicates that using higher order elements is not necessary.

In the drilling patterns study, the results show that the five-spot and the nine-spot patterns give the highest production rates for any number of wells. They are therefore the best patterns among the four patterns studied to be used to develop a new field.

This ability to predict the production rates of the wells in a reservoir allow the operator to be able to identify those wells that are performing below expectation and subject them to remedial or work-over operations.

References


SI Metric Conversion Factors

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<th>English Unit</th>
<th>Conversion Factor</th>
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