Double Approximation Methods for Some Boundary Integral Equations

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Abstract

We consider the problem of analyzing the convergence rate of projection methods for solving a variety of integral equations which arise from integral reformulations of boundary value problems when both integration and data approximation errors are present. This is done by extending Miel’s perturbation theory for split equations of the first kind [6] and generalizes the work of Anselone and Lee [5]. To illustrate the theory we discuss the numerical solution of some boundary integral equations which arise from the conversion of boundary value problems for Laplace’s equation in the plane to equivalent boundary integral equations. In particular, we show that the dominant error comes from approximating the kernel and right-hand sides. Such errors appear to have been largely ignored in previous analyses. Although we discuss a restricted class of examples in this paper, our general results can be applied to many other situations in both 2D and 3D.

Keywords: Boundary integral equation, projection method; convergence rate; double approximation

1 Introduction

Over the past 10 years there have been a large number of articles devoted to the topic of discrete projection methods for solving a variety of boundary integral equations (bies) [1-4]. In this work one is primarily interested in the effect of numerical integration errors on the underlying projection method with the goal of determining the minimum number of nodes necessary to preserve the convergence rates of the exact algorithm. In essentially all of this work it has been assumed that the kernel and right-hand sides of the equation are evaluated exactly, something which is rarely the case. In fact, it has been pointed out by
Anselone and Lee in [5] that the integration and approximation errors can be of comparable magnitude. In this paper we extend the work in [3, 4] to simultaneously account for both data and numerical integration errors. This extends the work in [5] for one-dimensional Fredholm equations to apply to a wide variety of single and multivariable boundary integral equations. Our goal is to determine sufficient conditions on approximations to the data so that the error bounds in [3] are preserved. Roughly, we show that in order to do this, we need to use minimax approximations to the data, which may not be practically possible.

The paper is divided into four sections. In Section 2 we establish a perturbation theorem generalizing Miel’s result in [6]. This enables us to simultaneously analyze approximation and integration errors for a general class of operator equations. Specializing this, allows us to obtain convergence results for numerous Galerkin and collocation methods for a variety of bies.

In Section 3 we show how to apply our results to analyze the convergence of a trigonometric based Galerkin method for a class of Fredholm integral equations which arise from solving boundary value problems for Laplace’s equation on domains in the plane with a smooth boundary.

In Section 4 we briefly comment on further applications of our techniques. Details will be given elsewhere.

2. Double approximation methods

2.1 Discrete projection methods

A large number of bies can be expressed abstractly in operator form as

\[ Hu = Ku + f \]  

(1)

where \( H: X \rightarrow Y \) is a bounded linear operator from a Banach space \( X \) to a Banach space \( Y \). The operator \( K: X \rightarrow Y \) is assumed to be compact [4]. In addition, we assume that \( H^{-1}: Y \rightarrow X \) is bounded. If the only solution to the homogeneous equation \( Hu - Ku = 0 \) is 0, then it follows from the Fredholm alternative that (1) has a unique solution which we assume to be the case for the rest of the paper.

When \( X = Y \) and \( H = I \), the identity operator, (1) typically represents a Fredholm equation such as occurs when solving Laplace’s or the Helmholtz equation by a double layer potential [4]. When \( H \neq I \), (1) can represent Cauchy singular equations [4], hypersingular equations [4] or equations of the first kind which arise when using single layer potentials to solve bvps [7].

A typical class of numerical methods for solving (1) are projection methods [4]. For this, let \( \{X_n\} \) be a sequence of subspaces of \( X \) with \( \dim(X_n) = N(n) \equiv N \) with the approximation property in \( X \) [4]. If \( \{\varphi_j\}_1^N \) is a basis for \( X_n \), we approximate \( u \) by \( u_n \) where
form the residual $\delta_n = Hu_n - Ku_n - f$ and set $\delta_n$ to zero in some fashion. In Galerkin’s method we assume that there is an inner product $\langle \cdot, \cdot \rangle_Y$ on $Y$ and then $\left\{ a_j \right\}_{j=1}^N$ are obtained by setting $\langle \delta_n, \psi_k \rangle_Y = 0$, $1 \leq k \leq N$, for $\left\{ \psi_k \right\}_{j=1}^N$, a linearly independent set in $Y$. In collocation, we set $\delta_n(x_k) = 0$, $1 \leq k \leq N$, for $N$ distinct points in the domain of $u$. Abstractly, these methods (and others) can be expressed in the form

$$P_n \delta_n = 0$$

(4)

where $P_n$ is the orthogonal projection onto $Y_n = \text{span}\left( \left\{ \psi_k \right\}_{j=1}^N \right)$ in Galerkin’s method and $P_n$ is an interpolatory projection in the case of collocation [4]. Hence, $u_n$ satisfies

$$P_n Hu_n = P_n Ku_n + P_n f.$$  

(5)

When $H = I$, we assume that $\psi_k = \varphi_k$, $1 \leq k \leq n$, so that $P_n Hu_n = P_n u_n = u_n$ and (5) simplifies to

$$u_n = P_n Ku_n + P_n f.$$  

(6)

In other situations it is the case that $HX_n = Y_n$ [4] so again $P_n Hu_n = Hu_n$ and (5) becomes

$$Hu_n = P_n Ku_n + P_n f.$$  

(7)

Although other cases occur in practice, we assume that (7) holds for the remainder of the paper [4]. If $P_n g \rightarrow g$ for all $g \in R(K)$, the range of $K$, it follows that $u_n \rightarrow u$ and satisfies the error bound

$$\|u - u_n\|_X \leq c \|Hu - P_n Hu\|_Y.$$  

(8)

Using known approximation results it is possible to derive convergences rates from (8). Numerous results of this type may be found in [4].

In practice, the use of (7) requires one to evaluate integral transforms $Ku_n$ and inner products $\langle Ku_n, \psi_k \rangle_Y$. In general, these cannot be done analytically and some form of numerical integration needs to be used in order to obtain a practical algorithm.

Abstractly, such algorithms can often be expressed in the form

$$Hv_n = \pi_n K_n v_n + \pi_n \in X_n,$$  

(9)

where $K_n$ is a numerical integration operator which approximates $K$ and $\pi_n = P_n$ for collocation and $\pi_n$ is a suitable discrete projection for Galerkin’s method [4]. to analyze the convergence of $v_n$ it is often convenient to rearrange (9) in the form

$$Hv_n = P_n K_n v_n + R_n v_n + P_n f + r_n$$  

(10)

where $R_n v_n = \pi_n K_n v_n - P_n K_n v_n$, $r_n = \pi_n f - P_n f$ and $r_n = P_n f - P_n f = 0$ for collocation. If $\pi_n : Y_n \rightarrow Y_n$, then the perturbations $R_n : X_n \rightarrow Y_n$ and $r_n \in Y_n$. In general, $R_n$ and $r_n$ can be expressed in terms of numerical integration errors [4].

If we define
then it was shown in \[4, 6\] that if \( |f_t| \to 0 \) and \( |r_j| \to 0 \), \( n \to \infty \), that for all \( n \) sufficiently large that \( v^* \) exists, is unique and \( v_n \to u, n \to \infty \). In addition, we have the error estimate [4]

\[
\|u - v_n\|_X \leq c \{\|u - u_n\|_X + \|R_n\|_n + \|r_n\|_y\}.
\]

Again this assumes that the kernel of \( K \) and hence of \( K_n \) and \( f \) are evaluated without error. We now consider generalizing (12) to account for approximations in these quantities as well.

### 2.2 A double approximation theorem

We now assume that \( K \) and \( f \) in (1) are approximated by \( \hat{K} \) and \( \hat{f} \) respectively. This leads to a further approximation \( \hat{v}_n \) to \( u \) satisfying

\[
H\hat{v}_n = \pi_n \hat{K} v_n + \pi_n \hat{f} = \hat{A}_n \hat{v}_n + \pi_n \hat{f}.
\]

Similarly, we write (9) as \( Hv_n = A_n v_n + \pi_n f \). Following Anselone and Lee in [5] we refer to (12) as a double approximation method for (1).

To analyze the convergence of \( \hat{v}_n \) we have the following theorem.

**Theorem 1.** Assume all the foregoing assumptions on \( H,K,f,P_n \) and \( \pi_n \). In addition, assume that \( \|A_n - \hat{A}_n\| \to 0, n \to \infty \). Then for all \( n \) sufficiently large, \( n \geq n_0 \), \( \hat{v}_n \) exists and

\[
\|u - \hat{v}_n\|_X \leq \|u - v_n\|_X + \|v_n - \hat{v}_n\|_X
\]

where

\[
v_n - \hat{v}_n = (H - \hat{A}_n)^{-1} \left[ P_n \left( K - \hat{K} \right) v_n + \pi_n \left( f - \hat{f} \right) + (R_n - \hat{R}_n) v_n \right].
\]

**Proof.** Equation (14) follows from the triangle inequality and \( u - \hat{v}_n = (u - v_n) + (v_n - \hat{v}_n) \).

To obtain (15) observe first that it follows from Banach’s lemma and \( \|A_n - \hat{A}_n\| \to 0, n \to \infty \), that for all \( n \geq n_0 \) that \( (H - \hat{A}_n)^{-1} |_{\gamma_n \to \gamma} \) exists [4] so that \( \hat{v}_n = (H - \hat{A}_n)^{-1} \pi_n \hat{f} \). Also, \( v_n = (H - A_n)^{-1} \pi_n f \). Thus
Since $A_n^{-1} = \hat{A}_n$, it follows from (14) that (15) holds, as required.

3 Double approximation for a bie

As an application of the previous theory we consider the solution of the integral equation

$$u(x) = f(x) + \int_0^{2\pi} k(x,t)u(t)dt$$

(17)

where $f$ is periodic on $[0,2\pi]$ and $k(x,t)$ is periodic in $(x,t)$ on $[0,2\pi] \times [0,2\pi]$. It then follows that $u$ is periodic as well. Such equations typically arise as boundary integral equations for partial differential equations on smooth domains in $\mathbb{R}^2$, particularly bies for Laplace's equations [4, 8].

To solve (17) numerically we consider a Galerkin method using trigonometric approximations [1, 4]. Although it has become traditional in the BEM literature to use piecewise polynomial approximations [4, 9], when the boundary of the domain and the boundary data are smooth, then $f$ and $k$ in (17) are smooth as well and trigonometric approximations are generally more rapidly convergent [4]. In addition, although collocation and Nyström methods have been preferred over Galerkin's method, Galerkin's method can lead to sparse matrix representations which may be more efficient than these other techniques [4]. For simplicity in our analysis, we assume that $f$ and $k$ are $C^\infty$.

Let $X = Y = L^2[0,2\pi]$ be the space of square integrable functions on $[0,2\pi]$ with the inner product to

$$\langle f, g \rangle = \int_0^{2\pi} f(t)\overline{g(t)}dt$$

(18)

where $\overline{g}$ is the complex conjugate of $g$. As usual, $\|f\| = (\langle f, f \rangle)^{1/2}$.

To define our numerical method let $X_n = \text{span}\left\{e^{i\theta} / \sqrt{2\pi}\right\}_{\theta \in [0,2\pi]}$ be the space of trigonometric polynomials of degree $\leq n$. As is well known, $\{X_n\}$ have the approximation property in $X$ [10]. Define the integral operator $K: X \to X$ by

$$Ku(x) = \int_0^{2\pi} k(x,t)u(t)dt.$$  

(19)

Since $k$ is $C^\infty$, $K$ is compact. Hence, (17) can be considered an operator equation of the form (1) in $X$ with $H = I$. 

$$v_n - \hat{v}_n = (H - A_n)^{-1} \pi_n f - (H - \hat{A}_n)^{-1} \pi_n \hat{f}$$

$$= (H - A_n)^{-1} \pi_n f - (H - \hat{A}_n)^{-1} (\pi_n \hat{f} + \pi_n f - \pi_n \hat{f})$$

$$= \left[ (H - A_n)^{-1} - (H - \hat{A}_n)^{-1} \right] \pi_n f + (H - \hat{A}_n)^{-1} (\pi_n f - \pi_n \hat{f})$$

$$= (H - \hat{A}_n)^{-1} (A_n - \hat{A}_n) (H - A_n)^{-1} \pi_n f + (H - \hat{A}_n)^{-1} \pi_n (f - \hat{f})$$

$$= (H - \hat{A}_n)^{-1} \left[ (A_n - \hat{A}_n) \pi_n + \pi_n (f - \hat{f}) \right].$$

Since $A_n - \hat{A}_n = P_n(K - \hat{K}) + R_n - \hat{R}_n$, it follows from (14) that (15) holds, as required.
To approximate \( u \) let \( u_n(x) = \sum_{j=-n}^{n} a_j e_j(x) \) (\( e_j(x) = \exp(2\pi i j x) \)) with \( \{a_j\}_{-n}^{n} \) determined by solving
\[
 a_k - \sum_{j=-n}^{n} a_j \langle Ke_j, e_k \rangle = \langle f, e_k \rangle, \quad -n \leq k \leq n. \tag{20}
\]
By definition,
\[
 \langle f, e_k \rangle = \int_{0}^{2\pi} f(t) e_k(t) dt \tag{21}
\]
and
\[
 \langle Ke_j, e_k \rangle = \int_{0}^{2\pi} \int_{0}^{2\pi} k(x,t) e_j(t) \overline{e_k(x)} dx dt. \tag{22}
\]
Letting \( P_n \) be the orthogonal projection onto \( X_n \), \( u_n \) satisfies
\[
 u_n = P_n Ku_n + P_n f. \tag{23}
\]
Since \( P_n g \to g \), for all \( g \in X \), it follows from our previous remarks that for all \( n \) sufficiently large that (23) (hence (20)) has a unique solution \( u_n \to u \) and \( \|u - u_n\| \leq c\|u - P_n u\| \). Since \( u = Ku + f \in C^\infty \), it follows from Jackson’s theorem for periodic functions and \( \|P_n\| = 1 \) that \( |4, 10| \)
\[
 \|u - u_n\| \leq cn^{-r}, \quad r > 0, \tag{24}
\]
where \( c \) depends on \( u \) but not on \( n \). Hence \( u_n \) converges faster than any power of \( 1/n \), in contrast to the algebraic convergence of piecewise polynomial approximations [4].

However, for a practical algorithm one generally has to approximate the integrals in (21)-(22). It is desirable to do this so that rapid convergence is preserved using a minimum of nodes. Since the trapezoidal rule converges rapidly for periodic functions, we consider the integration rules
\[
 Q_{2n+1}(f) = \frac{2\pi}{2n+1} \sum_{p=0}^{2n} f(x_p), \quad x_p = \frac{2\pi p}{2n+1}, \quad 0 \leq p \leq 2n, \tag{25}
\]
and approximate
\[
 \langle f, e_k \rangle \approx Q_{2n+1} \langle f e_k \rangle, \quad -n \leq k \leq n, \tag{26}
\]
and
\[
 \langle Ke_j, e_k \rangle \approx Q_{2n+1} \times Q_{2n+1} \langle ke_j e_k \rangle \tag{27}
\]
where \( Q_{2n+1} \times Q_{2n+1} \) is the product rule corresponding to \( Q_{2n+1} \). An important property of \( Q_{2n+1} \times Q_{2n+1} \) for our purposes is that \( Q_{2n+1}(f) = \int_{0}^{2\pi} f(t)dt \) for all \( f \in X_{2n} \) (i.e., the trapezoidal rule is essentially Gaussian for periodic functions). Using (26)-(27) to approximate the integrals in (21)-(22) we obtain the *discrete Galerkin approximation* \( v_n \) to \( u \) as \( v_n(x) = \sum_{j=-n}^{n} b_j e_j(x) \) where \( \{b_j\}_{-n}^{n} \) satisfy
\[
 b_k - \sum_{j=-n}^{n} b_j Q_{2n+1} \times Q_{2n+1} \langle ke_j e_k \rangle = Q_{2n+1} \langle f e_k \rangle, \quad -n \leq k \leq n. \tag{28}
\]
The convergence of \( v_n \) was discussed in [4]. Here, we briefly review those results.
Defining the discrete projection (Sloan in [11] refers to this as an \textit{hyperinterpolation operator})

\[ \pi_n u(x) = \sum_{j=-n}^{n} Q_{2n+1}(u\tilde{e}_j)e_j \]  

(29)

and the discrete integral operator \( K_n \) by

\[ K_n u(x) = \frac{2\pi}{2n+1} \sum_{j=-n}^{n} k(x,t_p)u(t_p) \]

(30)

it can be shown that \( u_n \) satisfies (10) with \( H = I \). Rearranging as in (10) we find that

\[ R_n v_n = -\sum_{j=-n}^{n} E_j(kv_n e_j) \]

(31)

and

\[ r_n = -\sum_{j=-n}^{n} e_j(f e_j)e_j \]

(32)

where \( \{E_j\} \) are the quadrature errors in approximating \( \langle Kv_n, e_j \rangle \) by \( Q_{2n+1} \times Q_{2n+1} \langle kv_n, e_j \rangle \) and \( \{e_j\} \) are the errors in approximating \( \langle f, e_j \rangle \) by \( Q_{2n+1} \langle f e_j \rangle \). From Ch. 5 of [4] it follows, using the Gaussian property of \( Q_{2n+1} \), that \( |E_j| \leq cn^{-r} \), \(-n \leq j \leq n\) and \( |e_j| \leq cn^{-r} \), \(-n \leq j \leq n\) for all \( r > 0 \), where the \( c \)'s depend on \( k \) and \( f \) but not on \( n \). Since \( \{e_j\}^n_{-n} \) are orthogonal in \( X \) we have

\[ \|R_n w_n\|^2 = \sum_{j=-n}^{n} |E_j|^2 \leq (2n+1)n^{-r} \leq cn^{-r+1}, \]

(33)

so that \( \|R_n w_n\| \leq cn^{-r+1/2} \forall w_n \in X_n \). Hence, from (11) \( \|R_n\| \leq cn^{-r+1/2} \). Since \( r \) can be chosen arbitrarily large, \( \|R_n\| \rightarrow 0, n \rightarrow \infty \).

Similarly, \( \|r_n\| \leq cn^{-r+1/2} \rightarrow 0, n \rightarrow \infty \). From Section 2 it now follows that \( v_n \rightarrow u \) and \( \|u - v_n\| \leq c(n^{-r} + n^{-r+1/2}) \leq cn^{-r+1/2} \). Since \( r \) can be chosen arbitrarily large, the discrete Galerkin approximation converges at essentially the rate as \( u_n \).

We now consider the double approximation method for \( u \) obtained by approximating the kernel \( k \) in (17) by \( f \). Such approximations arise naturally when one approximates the boundary and boundary data in the given boundary value problem leading to (17). This leads to the approximation \( \hat{u}_n \) for \( u \) satisfying (10) where \( \hat{K} \) is the integral operator (19) with \( k \) replaced by \( \hat{k} \) and \( \hat{K}_n \) with \( k \) replaced by \( \hat{k} \) as well.

To establish convergence of \( \hat{u}_n \) to \( u \) we use Theorem 1. To do this we first need to establish that \( \|A_n - \hat{A}_n\| \rightarrow 0, n \rightarrow \infty \). Now \( A_n - \hat{A}_n = P_n(K - \hat{K}) + R_n - \hat{R}_n \) so that \( \|A - \hat{A}_n\| = \|P_n(K - \hat{K})\| + \|R_n - \hat{R}_n\| \leq \|P_n(K - \hat{K})\| + \|R_n - \hat{R}_n\| \).
Since \( P_n \) is an orthogonal projection, \( \|P_n\| = 1 \) so that \( \|P_n(K - \hat{K})\| \leq \|K - \hat{K}\|/\sum_{j=-n}^{n} E_j(kw_e e_j) - \sum_{j=-n}^{n} E(\hat{kw}_e e_j) \) so that \( \|R_n - \hat{R}_n\| \leq c_2 n^{r+1/2} \) for all \( r > 0 \). Hence, \( \|R_n - \hat{R}_n\| \leq cn^{r+1/2}, r > 1/2 \). In addition, if \( \hat{k} \) is chosen so that \( \|k - \hat{k}\| < cn^{-r'}, r' > 0 \), then \( \|P_n(K - \hat{K})\| \leq cn^{-r'} \).

Thus, \( \|A_n - \hat{A}_n\| \leq cn^{-r'}, r' > 0 \) so that \( \|A_n - \hat{A}_n\| \rightarrow 0, n \rightarrow \infty \). From Theorem 1, \( \hat{v}_n \) exists for all \( n \) sufficiently large and (15) holds.

From (14)-(15) we obtain the bound \( \|u - \hat{v}_n\| \leq \|u - v_n\| + c\|K - \hat{K}\|\|v_n\| + \|A - \hat{A}_n\|\|v_n\| + \|\pi_n(f - \hat{f})\| \). Since \( \|v_n\| \) is uniformly bounded, it suffices to bound \( \|\pi_n(f - \hat{f})\| \).

**Lemma 1.** Let \( \pi_n \) be defined by (29). If \( g \) is continuous, then \( \|\pi_n g\| \leq 2\pi(2n + 1)^{1/2} \|g\| \). \( (34) \)

**Proof.** By definition, \( \pi_n g = \sum_{j=0}^{n} Q_{2n+1}(ge_j) e_j \). Since \( \{e_j\}_{-n}^{n} \) are orthonormal, \( \|\pi_n g\|^2 = \sum_{j=-n}^{n} Q_{2n+1}^2(ge_j) \). But \( Q_{2n+1}(ge_j) \leq [2\pi/(2n + 1)] \sum_{j=0}^{2n} |g(x_j)| \leq 2\pi \|g\| \), since \( |e_j| \leq 1 \). Thus, \( \|\pi_n g\|^2 \leq 4\pi^2 (2n + 1) \|g\|^2 \) so that \( \|\pi_n g\| \leq 2\pi(2n + 1)^{1/2} \|g\| \) as required. \( \square \)

From Lemma 1, \( \|\pi_n(f - \hat{f})\| \leq cn^{1/2} \|f - \hat{f}\|_\infty \). If \( \|f - \hat{f}\|_\infty \leq cn^{-r'}, r' > 1/2 \) we get \( \|u - \hat{v}_n\| \leq c_n^{r'+1/2} + c_2 n^{r' - 1/2} \).

From the above argument we see that the error in the double approximation \( \hat{v}_n \) is dominated by the errors in \( k - \hat{k} \) and \( f - \hat{f} \). Hence, in order to retain the optimal convergence rates of \( u_n \) and \( v_n \), \( \hat{k} \) and \( \hat{f} \) must be chosen so that these errors are \( O(n^{-r}) \) where \( r \) is arbitrarily large. By Jackson’s theorem they need to be of the order of the minimax approximations (i.e. best uniform approximation) to \( k \) and \( f \) respectively [10]. In general, this will not be feasible.

For example, a simple way to approximate \( f \) is to let \( \hat{f} \) be the interpolant to \( f \) on \( \{x_p\}_{0}^{2n} \). As is well known, \( \hat{f}(x) = \sum_{p=0}^{2n} D_p(x) f(x_p) \) where \( D_p(x_m) = \delta_{pm} \) [4, 10] and \( \{D_p\} \) do not depend on \( f \). It is also known that \( \|f - \hat{f}\|_\infty \leq (c \log n) n^{-r} \) if \( f \) is \( C^r \) [4, 10]. However, to compute \( \hat{f} \) we need to know \( \{f(x_p)\}_{0}^{2n} \) exactly. If this is not the case, then \( \hat{f} \) will not satisfy the required conditions. Similar remarks apply to \( k \). In fact, if \( \{f(x_p)\}_{0}^{2n} \) and \( \{k(x_p, t_m)\}_{0}^{2n} \) are known exactly,
then \( v_n \) can be computed without error in \( k \) and \( f \) as follows from (29), so there will be no need to do a further approximation. This argument suggests that in most situations it will not be possible to obtain optimal convergence rates for \( \hat{v}_n \).

4 Further Applications

Although we have focused on equations of the second kind, Theorem 1 can be applied to many other boundary integral equations. For example in [4] we considered discrete projection methods for boundary integral equations of the first kind which arise when using single layer, rather than double layer representations for Laplace’s equations. Similarly, we can analyze Cauchy singular and hypersingular equations which arise when studying fluid flows around airfoils and hydrofoils. Equations on surfaces in \( \mathbb{R}^3 \) using spherical harmonic approximations can also be dealt with in a similar fashion [2-4]. To deal with piecewise polynomial approximations one can use the theory developed by Atkinson and Bogomolny in [1]. These possibilities will be treated in future work.

Last, we note that collocation and Nyström methods can also be handled using the techniques of this paper.

References


