Solution of Electrical Impedance Tomography Equations Using Boundary Element Methods *

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Abstract

Imaging using Electrical Impedance Tomography requires repeated solution of the diffusion equation \( \nabla \cdot (\sigma \nabla \phi) = 0 \), for the electric potential \( \phi \) for successive approximations of the spatially varying conductivity, \( \sigma \). The dual reciprocity boundary element method is an attractive method for the efficient solution of the problem, and the feasibility of using such a method was shown earlier by us. In this article we attempt to optimize the method on an example problem by studying the influence of using different interpolating radial basis functions, and of the distribution of internal points on the accuracy of the solution.

1 Introduction

In electrical impedance tomography (EIT) the distribution of impedances inside a volume is sought by applying specified currents at some surface electrodes, and performing measurements of the voltage at other electrodes. The equations for the electric field then provide a relationship between the impedance distribution inside the medium and the measured voltages and applied currents. Obtaining the distribution of the electric potential given the applied currents and the impedance distribution is the forward problem of EIT, while recovering the impedance distribution knowing the current and electric potential constitutes the inverse problem of EIT. This inverse problem is solved via an iterative minimization procedure in which a sequence of forward problems with improved guesses for the impedance distribution are solved. Earlier we have shown the feasibility of achieving solutions using “dual reciprocity” [1] BEM [2].

However, to optimize the method for practical problems several issues related to the use of these methods must be resolved. The use of dual reciprocity methods for the solution of partial differential equations is an area where practice has gone beyond the theory, and lacking formal convergence proofs, practitioners must use ‘experimental’ evidence to make choices to optimize algorithms. Our numerical experiments examine the influence of three factors on the accuracy of our three-dimensional dual-reciprocity BEM codes.

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First, we test the influence of the choice of the type of radial basis functions. These include functions of the form proposed in Ref. [1], three-dimensional Gaussian blobs used in vortex methods [3], and multiquadric functions [4]. Expressions for associated functions required in the dual-reciprocity method are also obtained.

Second, using a result proven by Powell for a two-dimensional problem [5] as a guideline, we add to each of the tested basis functions a small number of global (i.e., depending upon the global coordinates) polynomial functions. Powell showed that interpolation with thin plate splines with the addition of a small number of polynomial functions, is convergent. In our analysis we restrict the polynomial set to the constant and linear terms.

Third, as we will see below, the dual-reciprocity method may be viewed as a grid-free method for integrating the domain term arising in a Green’s function formulation of an inhomogeneous equation. In the field of multi-dimensional Monte-Carlo integration it is well-known that, on average, most accurate sampling of the domain is provided by integration points based on Sobol quasi-random sequences [6]. In the dual-reciprocity method a desideratum is to use the minimum number of internal collocation points. Accordingly we investigate the use of such internal collocation points and compare it with one based on a structured set of internal collocation points.

2 Governing Equations

Let us consider an Electrical Impedance Tomography problem where we know the current at all of the boundary $S$ of a domain $\Omega$, and the voltage at selected points on the boundary. The current in between the electrodes is taken to be zero. The electrical potential at the electrodes is also available.

The electric potential, $\phi$, satisfies the following equation where $n$ is the boundary normal [7]:

$$\nabla \cdot (\sigma \nabla \phi) = 0 \quad \text{in} \quad \Omega$$

subject to

$$\begin{cases}
\sigma \frac{\partial \phi}{\partial n} \quad \text{and} \quad \phi \text{ known at the electrodes} \\
\frac{\partial \phi}{\partial n} = 0 \quad \text{on the rest of the boundary}.
\end{cases}$$

A direct method for obtaining $\sigma$ from such measurements is not readily available. Instead, starting from a guessed distribution of $\sigma$, a ‘forward problem’ is solved. Then, minimization of the error between the predicted and the measured values of $\phi$ on the boundary is sought for the next guess of the $\sigma$ distribution, and the procedure is repeated until satisfactory convergence is achieved. As a part of the solution procedure the above equations must be solved repeatedly.

2.1 Analytical Solution

In what follows we wish to compare the different approaches on a particular test problem. Accordingly we will restrict our application to the following exact solution. Considering a distribution of $\phi$ that depends only on $x$ and that satisfies Equation (1), we have

$$\sigma_x \phi_x + \sigma \phi_{xx} = 0.$$
We look for solutions of the form \( \phi = \exp(f(x)) \), and solving for \( \sigma \):

\[
\sigma = \exp(-\log(f_x) - f) = \frac{\exp(-f)}{f_x}.
\] (4)

This defines a family of solutions for different choices of \( f \). Substituting for \( \phi \) and its derivatives we select the following test solution for checking the code

\[
f(x) = x, \quad \phi = e^x, \quad \sigma = e^{-x}.
\] (5)

We consider now a three-dimensional spherical geometry body as the domain of application of the EIT problem, and use the distribution given in (5) as the exact solution that will be used to check our three dimensional code.

3 Dual Reciprocity BEM for EIT

3.1 Alternative forms of the governing equations

For convenience of the BEM formulation, Equation (1) is expressed in one of two forms which explicitly display the Laplacian operator in the governing equations. In the first, the equation is represented as

\[
\nabla^2 \phi = -\nabla \log \sigma \cdot \nabla \phi = b(x, \phi).
\] (6)

In this form the equation is subject to the same boundary conditions as in (2).

The EIT problem can also be formulated as Helmholtz equation by making the variable transformation, \( u = \sqrt{\sigma} \phi \), which leads to

\[
\nabla^2 u = \kappa u, \quad \text{where} \quad \kappa(x) = \frac{\nabla^2 \sqrt{\sigma}}{\sqrt{\sigma}}.
\] (7)

To complete the change of variable the boundary conditions are also transformed, with a general linear boundary condition being converted from

\[
\alpha \phi + \beta \frac{\partial \phi}{\partial n} = f, \quad \text{to} \quad \left( \alpha - \beta \frac{n \cdot \nabla \sigma}{2\sigma} \right) u + \beta \frac{\partial u}{\partial n} = \sqrt{\sigma} f.
\] (8)

This transformation includes the Dirichlet and Neumann conditions of (2).

In Ref. [8] we showed that Dual-Reciprocity BEM (DRBEM) algorithms based on Equation (7) were more accurate, as they did not require interpolation of the derivatives of \( \phi \). Accordingly in what follows we will use this form of the governing equation in our algorithms.

3.2 Green’s Identity

Let us denote the fundamental solution to Laplace’s equation by \( G \), so that

\[
\nabla^2 G(x, y) = 4\pi \delta(x - y), \quad \text{where} \quad G = -|x - y|^{-1}.
\] (9)

As presented below Equation (1) can be reformulated via Green’s identity:

\[
a\pi \phi(x) = \int_{\Omega} \nabla^2 \phi(y) G(x, y) dV + \int_S n \cdot [\phi(y) \nabla G(x, y) - G(x, y) \nabla \phi(y)] dS,
\] (10)

where \( a\pi \) is the solid angle under which the point \( x \) sees the rest of the domain. The surface integrals are performed by suitably discretizing the boundaries, using
plane triangular discretizations (see Figure 1). Over each boundary element $S_k$ a linear Lagrangian interpolation of $\phi$ and $\partial \phi / \partial n$ is performed using the values at the nodes (triangle vertices). For Laplace’s equation the volume integral in Equation (10) vanishes, which, on accounting for boundary conditions, provides a closed system of equations, which may be solved for $\phi$ and $\partial \phi / \partial n$ at the boundary, and $\phi$ at any other point $x$.

For equations of the type

$$\nabla^2 \phi = b(x, \phi, \nabla \phi),$$  \hspace{1cm} (11)

substituting for $\nabla^2 \phi$ in Equation (10) leads to

$$a \pi \phi = \int_S \left( \phi \frac{\partial G}{\partial n} - G \frac{\partial \phi}{\partial n} \right) dS + \int_{\Omega} G b d\Omega. \hspace{1cm} (12)$$

The presence of a non constant $b$ in the domain integral prevents a BEM formulation. To overcome this difficulty and transform Equation (12) into a BEM formulation the Dual Reciprocity (DR) method [1] is applied. To do so $b$ is expressed in terms of a special set of known functions $\{f_j\}$, $j = 1, \ldots, N$.

$$b(x) = \sum_{j=1}^{N} \alpha_j f_j(x), \hspace{1cm} b = F \alpha, \hspace{1cm} \alpha = F^{-1} b. \hspace{1cm} (13)$$

For each function $\{f_j\}$ there exists another known function $\{\psi_j\}$ related to it by

$$\nabla^2 \psi_j = f_j,$$  \hspace{1cm} (14)

i.e., $\psi_j$ is a particular solution with $f_j$ as the right hand side. We can use Green’s identity (10) for $\psi_j$ and express the domain integral in (12) as

$$\sum_{j=1}^{N} \alpha_j \int_{\Omega} f_j G dV = \sum_{j=1}^{N} \alpha_j \int_{\Omega} \nabla^2 \psi_j G dV = \sum_{j=1}^{N} \alpha_j \left[ a \pi \psi_j - \int_{S} \mathbf{n} \cdot (\psi_j \nabla G - G \nabla \psi_j) dS \right]. \hspace{1cm} (15)$$

By substituting in (12) we obtain the following formulation integrals on the boundary only:

$$a \pi \phi = \int_S \left( \frac{\partial G}{\partial n} \phi - G \frac{\partial \phi}{\partial n} \right) dS + \sum_{j=1}^{N} \alpha_j \left[ a \pi \psi_j - \int_{S} \left( \frac{\partial \psi_j}{\partial n} - \psi_j \frac{\partial G}{\partial n} \right) dS \right]. \hspace{1cm} (16)$$

Again discretizing the boundaries using $K$ nodes and collocating, the following equation is obtained:

$$a \pi \phi(x_i) = \sum_{k=1}^{K} \left[ B_{ik} \phi_k - A_{ik} \frac{\partial \phi_k}{\partial n} \right] + \sum_{j=1}^{N} \alpha_j \left[ c \pi \psi_j(x_i) + \sum_{k=1}^{K} \left( B_{ik} \psi_{kj} - A_{ik} \frac{\partial \psi_{kj}}{\partial n} \right) \right], \hspace{1cm} (17)$$

where $i = 1, \ldots, K$. It can be expressed in matrix vector form as

$$B \phi - A \frac{\partial \phi}{\partial n} = (B \psi - A \frac{\partial \psi}{\partial n}) F^{-1} b. \hspace{1cm} (18)$$
Here \( \mathbf{A} \) and \( \mathbf{B} \) are matrices corresponding to the discretization and integration with the Green's function, \( G \), and its normal derivative.

In the EIT problem the term \( \mathbf{b} \) depends on the unknown function to be determined, \( \phi \). By using (13) the right hand side \( \mathbf{b} \) in (6) can be interpolated as:

\[
\mathbf{b} = [k \mathbf{F}^{-1}] \phi = \mathbf{H} \phi,
\]

where \( k \) is a vector containing the values of \( k = \nabla^2 \sqrt{\sigma} / \sqrt{\sigma} \) at the points at which the domain term is collocated, and the term in square brackets is a term by term product. Substituting for \( \mathbf{b} \) Equation (18) becomes

\[
\mathbf{A} \frac{\partial \phi}{\partial \mathbf{n}} = [\mathbf{B} - \mathbf{S}(k)] \phi, \quad \text{where} \quad \mathbf{S} = \left( B \psi - A \frac{\partial \psi}{\partial \mathbf{n}} \right) \mathbf{F}^{-1} \mathbf{H}. \quad (20)
\]

Our tests with the DRBEM code developed (described in [2]) showed that when \( \mathbf{b} \) depends on \( \phi \) and/or its derivatives the accuracy of the solution suffers if \( \mathbf{b} \) or \( \phi \) show considerable variation inside the problem domain. In these cases we have found that the addition of a few internal points improves significantly the accuracy of interpolation of the right hand side and enables the method to become very accurate, as is necessary for its use in EIT. Therefore, we add \( K_I \) internal collocation points to the \( K \) boundary collocation points. Collocating the equation at the \( K \) boundary points yields,

\[
an \phi(x_i) = \sum_{k=1}^{K} \left[ B_{ik} \phi_k - A_{ik} \frac{\partial \phi_k}{\partial \mathbf{n}} \right] + \sum_{j=1}^{K+K_I} \left[ c \tau \psi_j(x_i) + \sum_{l=1}^{K} \left[ -A_{il} \frac{\partial \psi_l}{\partial \mathbf{n}} + B_{il} \psi_l \right] \right] H_{jk} \phi_k.
\]

(21)

To close the system of equations we need \( K_I \) additional equations for the new \( \phi_k \). This can be obtained by writing Green's identity with \( x \) taken to be at the internal points. Collocating at the \( K_I \) internal points we obtain equations formally similar to Equation (21), except that the quantity \( a \) takes on the value appropriate to an internal point (\( 4\pi \) in 3D). We must emphasize that the number of internal nodes that are needed is quite small — at most of the same order as the number of boundary nodes. Further, these internal nodes need not be part of a regular grid, i.e. they can be distributed in an "unstructured" fashion. Thus problems associated with mesh generation will not be encountered.

### 3.3 Choice of Expansion Functions

In the above we assumed that functions for performing the dual-reciprocity expansion, \( f_j \), are available. In earlier work we used a set of functions proposed by Partridge et al [1]. These functions are based on the distance between the collocation point and the point of interest, so that

\[
f_j(x) = 1 + |x_j - x|.
\]

(22)

This set of functions is already computed in the BEM algorithm since the Green's function involves \( |x_j - x| \). These functions are termed radial functions as they depend only on the radial distance between a point, the center of the function \( x_j \), and the evaluation point \( x \). They can thus be defined in terms of

\[
r = \sqrt{(x - x_j)^2 + (y - y_j)^2 + (z - z_j)^2}.
\]
Good practical convergence properties were demonstrated for these functions [1]. Recently the connection between these functions and the radial basis functions, an area of active research area in theoretical numerical analysis [4, 10, 5], has been established.

Here we will consider use of these radial basis functions in the dual reciprocity method. The functions are given in the table below. Use of a function $f_j$ in the dual-reciprocity method also requires knowledge of the associated functions $\psi_j$. In three dimensions the associated functions can be calculated by integrating

$$\frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial \psi_j}{\partial r} \right) = f_j$$

(23)

These associated functions are also provided in the table.

<table>
<thead>
<tr>
<th>Function</th>
<th>$f_j$</th>
<th>$\psi_j$</th>
</tr>
</thead>
<tbody>
<tr>
<td>constant</td>
<td>1</td>
<td>$r^2/6$</td>
</tr>
<tr>
<td>linear</td>
<td>$r$</td>
<td>$r^3/12$</td>
</tr>
<tr>
<td>quadratic</td>
<td>$r^2$</td>
<td>$r^4/20$</td>
</tr>
<tr>
<td>Gaussian</td>
<td>$\exp\left(-r^3/c^3\right)$</td>
<td>$\frac{c^3}{3r^2} \left(1 - e^{-r^3/c^3} + \frac{r^3}{c^3} \left(\frac{3}{2} \frac{r^3}{c^3}\right)\right)$</td>
</tr>
<tr>
<td>multiquadric</td>
<td>$\sqrt{r^2 + c^2}$</td>
<td>$\frac{1}{4} \left(r^2 + \frac{3}{8} c^2\right) \frac{1}{r} \log\frac{r + \frac{c}{2}}{\frac{c}{2}} - \frac{c^2}{3}$</td>
</tr>
<tr>
<td>monomial</td>
<td>$x_{i,i}$, $i = 1, 2, 3$</td>
<td>$x_i^3/6$</td>
</tr>
</tbody>
</table>

It should be noted that the Gaussian function has been extensively used in the vortex element method literature, and convergence proofs have been established for the solution of the Euler equations of fluid mechanics using such functions [3]. Interestingly this large body of work in the field of vortex elements does not seem to have come to the attention of the dual-reciprocity BEM community.

3.3.1 Modified radial basis functions

In Ref. [5] convergence of the expansion of a function in terms of radial basis functions was proved when a small number of global polynomials is added to the basis. In particular it was shown that in two-dimensions expansions of the form

$$b(x, y) = \sum_{j=1}^{N} \alpha_j f_j (|x - x_j|) + \alpha_{N+1} + \alpha_{N+2} x + \alpha_{N+3} y,$$

(24)

with the coefficients $\alpha_j$ obtained by collocating at the $N$ centers of the functions $x_j$, along with satisfying the following three constraint relations

$$\sum_{j=1}^{N} \alpha_j = 0, \quad \sum_{j=1}^{N} \alpha_j x_j = 0, \quad \sum_{j=1}^{N} \alpha_j y_j = 0,$$

(25)

converge uniformly to the underlying function. Here we will extend this definition to 3D, by adding the monomial $\alpha_{N+4} x$ to Equation (24) and by adding a corresponding constraint equation to (25).

We also note that in Ref. [5] it was shown that the interpolant resulting from this procedure is the interpolant that minimizes curvature if the radial basis function $f_j (|x - x_j|)$ is chosen to be the Green’s function of the biharmonic operator, $r^2 \log r$. The Green’s function for the biharmonic operator in 3D is the familiar function used in dual reciprocity methods, $1 + r$. This could explain the robustness of dual-reciprocity methods based on this function.
3.4 Distribution of Inner Points

For a given geometry if one has a sufficiently accurate discretization on the domain boundary, and the distribution of inner points is also fixed, much of the work involved in the present algorithm need only be performed once. Thus, if there is a good way to generate an internal distribution of points that sample the internal domain in a fashion that ensures that contributions from the domain will be well resolved we would have a very efficient algorithm.

Precisely such a distribution of points, the so-called Sobol points, are used in multi-dimensional Monte Carlo integration [6], and in optimal sampling [11]. These points satisfy a condition that the ratio of the number of points in any sub-domain to the total number of points in the domain is equal to the ratio of the volume of the sub-domain to the volume of the entire domain. Such a condition cannot be satisfied by regular grids. The Sobol points have been known to achieve orders of magnitude more accurate results than other sampling points. The generation of these points is a complicated process. However efficient subroutines for their generation are available in Ref. [6] and in the public-domain software repository netlib [12], which we used here.

4 Numerical Experiments and Results

4.1 Procedure

We solved Equation (7) on a spherical domain using our developed software package 3DynaEIT. The problem considered has the exact solution given in Equation (5). The gridding for the surface of the sphere was performed using 146 nodes and 288 panels. Such a discretization has been shown to be quite accurate for BEM calculations of Laplace’s equation in Ref. [13].

The domain inside is discretized using two alternate discretizations. The first corresponds to the distribution of an even number of points along each of the coordinate axes, along the bisectors of the axes in each of the coordinate planes, and along the diagonals of the inscribed cube that has its faces parallel to the coordinate planes. Five cases with 26, 52, 78, 130 and 260 points were considered.

The second discretization corresponds to the use of Sobol points. Here eight cases with 2, 4, 8, 16, 32, 64, 128 and 256 points were considered. Also for comparisons we used computations without internal points and with one point placed into the center of the domain.

The equations were solved using the four families of radial basis functions described earlier. Additionally the approximations were performed using expansions of the types in Equation (13) and (24) respectively. Two of the radial basis functions, the Gaussian and the multiquadric, contain adjustable parameters. In each case the equations were solved for twenty values of the respective parameter from the segment [0.01,2].

For estimations of the accuracy of the procedure the following two measures for each solution was computed

\[
\epsilon_{\text{max}} = \max_i \left\{ \frac{|\phi_i - \phi_i^{(e)}|}{\phi_j^{(e)}_{\text{max}} - \phi_j^{(e)}_{\text{min}}} \cdot \frac{|\partial\phi_i/\partial n - \partial\phi_i^{(e)}/\partial n|}{\partial\phi_j^{(e)}/\partial n_{\text{max}} - \partial\phi_j^{(e)}/\partial n_{\text{min}}} \right\}
\] (26)
Here \( \phi^{(e)} \) is the exact solution given above. The first measure shows the maximum relative error of calculations, while the second one shows the relative error averaged over all nodes.

### 4.2 Results

First, we found that both average and maximum errors depend on the parameter \( c \) in the Gaussian and multiquadric functions in an irregular fashion, and show the need for more theoretical work with the convergence of expansions with these functions. Accordingly we selected for comparisons the best results with minimum \( \epsilon_{max} \).

Second, we discovered that the number of internal points for most cases does not much influence \( \epsilon_{max} \) (see Tables 1 and 2). Both average and maximum error decrease substantially (by up to one order of magnitude) with the addition of the first internal point. The gain in accuracy due to the addition of further internal collocation points compared to the expense of computation with the increased number of internal points is insignificant. If the number of internal points is comparable with the number of surface nodes (cases 256 and 260), then maximum and average errors substantially increase in some cases. This is especially clear for the regular point distribution (see Table 1), where the maximum error does not depend even on the type of the radial basis function.

### Table 1: Dependence of the maximum relative error \( \epsilon_{max} \) (%) on the type of radial basis function and the number and distribution of internal nodes. R indicates a regular grid, while S indicates a grid based on Sobol points. The notation (+g) indicates the modified radial basis functions.

<table>
<thead>
<tr>
<th>Functions</th>
<th>R/S (0)</th>
<th>R/S (1)</th>
<th>R (26)</th>
<th>S (16)</th>
<th>R (260)</th>
<th>S (256)</th>
</tr>
</thead>
<tbody>
<tr>
<td>( 1 + r )</td>
<td>27</td>
<td>5.3</td>
<td>5.5</td>
<td>5.8</td>
<td>6.1</td>
<td>6.2</td>
</tr>
<tr>
<td>( 1 + r + r^2 )</td>
<td>24</td>
<td>1.8</td>
<td>1.9</td>
<td>2.1</td>
<td>6.1</td>
<td>2.2</td>
</tr>
<tr>
<td>( \exp(-\frac{r}{c^2}) )</td>
<td>13</td>
<td>5.4</td>
<td>1.1</td>
<td>2.5</td>
<td>6.1</td>
<td>2.2</td>
</tr>
<tr>
<td>( \sqrt{r^2 + c^2} )</td>
<td>21</td>
<td>1.9</td>
<td>2.2</td>
<td>3.2</td>
<td>6.1</td>
<td>4.5</td>
</tr>
<tr>
<td>( 1 + r (+g) )</td>
<td>18</td>
<td>1.8</td>
<td>1.6</td>
<td>2.1</td>
<td>6.1</td>
<td>2.2</td>
</tr>
<tr>
<td>( 1 + r + r^2 (+g) )</td>
<td>20</td>
<td>1.8</td>
<td>1.5</td>
<td>2.1</td>
<td>6.1</td>
<td>2.2</td>
</tr>
<tr>
<td>( \exp(-\frac{r}{c^2}) (+g) )</td>
<td>13</td>
<td>1.8</td>
<td>1.2</td>
<td>2.1</td>
<td>6.1</td>
<td>2.2</td>
</tr>
<tr>
<td>( \sqrt{r^2 + c^2} (+g) )</td>
<td>17</td>
<td>1.7</td>
<td>1.2</td>
<td>2.1</td>
<td>6.1</td>
<td>2.2</td>
</tr>
</tbody>
</table>

The influence of the internal point distribution type (regular or Sobol) on the error is small when the number of internal points is smaller than the number of
surface nodes. For larger numbers of internal points computations with Sobol
distributions exhibit lower maximum and relative error than ones with regular
distributions. At the same time for small and intermediate numbers the regular
distribution with some selected radial function and parameter can be better.
Computations with Sobol distributions show that the error in this case is less
sensitive to changes of the number of points and type of the radial basis function.
The last point is especially true for the modified basis functions (see the last four
functions in tables 1 and 2).

Comparing results obtained with regular and modified radial basis functions,
we notice that for small and moderate numbers we always have lower maximum
and average errors using modified functions, except only one case when the max-
imum error increases from 1.1 to 1.2%. For other cases the gain due to using
modified functions is substantial (see the function $1 + r$). When the number of
internal points is comparable with the number of surface nodes the difference
between results obtained with regular and modified functions is negligible.

A first look at the comparison between the four selected radial basis functions
for the minimum error can lead us to a conclusion that the worst function is
$1 + r$ and the best is the Gaussian function. It is true that these function give
the maximum and minimum of $\epsilon_{\text{max}}$ in the table. But if we look at modified
functions, we find that the difference between “the best” and “the worst” is not
large. Moreover, when using functions with parameters such as the regular multi-
quadric or Gaussian functions we have strong oscillations of the error (see Figure
2). Lacking a proper convergence criterion, it is difficult to make a guess for the
best value of the parameter. For modified functions with Sobol points we have a
more stable situation (see Figure 2). For stable predictable results it is better to
use a modified function without adjustable parameters to avoid the risk of large
error.

<table>
<thead>
<tr>
<th>Functions</th>
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<th>R (260)</th>
<th>S (256)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$1 + r$</td>
<td>2.7</td>
<td>0.19</td>
<td>0.17</td>
<td>0.19</td>
<td>0.097</td>
<td>0.090</td>
</tr>
<tr>
<td>$1 + r + r^2$</td>
<td>0.18</td>
<td>0.080</td>
<td>0.073</td>
<td>0.086</td>
<td>0.066</td>
<td>0.052</td>
</tr>
<tr>
<td>$\exp(-\frac{r^2}{2})$</td>
<td>0.26</td>
<td>0.24</td>
<td>0.037</td>
<td>0.068</td>
<td>0.058</td>
<td>0.040</td>
</tr>
<tr>
<td>$\sqrt{r^2 + c^2}$</td>
<td>0.18</td>
<td>0.089</td>
<td>0.078</td>
<td>0.12</td>
<td>0.063</td>
<td>0.066</td>
</tr>
<tr>
<td>$1 + r (+g)$</td>
<td>0.13</td>
<td>0.041</td>
<td>0.034</td>
<td>0.042</td>
<td>0.059</td>
<td>0.040</td>
</tr>
<tr>
<td>$1 + r + r^2(+g)$</td>
<td>0.15</td>
<td>0.047</td>
<td>0.040</td>
<td>0.048</td>
<td>0.060</td>
<td>0.042</td>
</tr>
<tr>
<td>$\exp(-\frac{r^2}{c^2})(+g)$</td>
<td>0.038</td>
<td>0.040</td>
<td>0.032</td>
<td>0.036</td>
<td>0.058</td>
<td>0.039</td>
</tr>
<tr>
<td>$\sqrt{r^2 + c^2}(+g)$</td>
<td>0.13</td>
<td>0.038</td>
<td>0.032</td>
<td>0.040</td>
<td>0.059</td>
<td>0.040</td>
</tr>
</tbody>
</table>

4.3 Conclusions

The results of our computations allow us to recommend dual reciprocity boundary
element methods for solution of EIT Equations. The accuracy of the numerical
solution and the required computation time depend on many factors including the type of the radial basis function and the number and distribution of internal nodes. Based on the analysis of more than 1000 variants of numerical solutions for the sphere with a fixed surface discretization, we found that:

1. The accuracy of the simulation strongly depends on the presence of internal nodes. Introducing a single internal collocation point reduces the error by an order of magnitude.

2. The optimal number of the internal collocation points is relatively small for this case. The best results are obtained when the number of internal points is a few percent of the number of surface nodes.

3. Substantial improvement of the accuracy is obtained by using modified radial basis functions rather than regular basis functions.

4. Use of Sobol representative sampling for internal points distribution did not result, in this test case, in a substantial gain in accuracy for the best case results. However, on average it made the error insensitive to variations of the radial basis function parameters. This stability is especially observed when the modified radial basis functions are employed.

5. The accuracy of the numerical solution depends on the chosen parameter in the case of some radial basis functions. For Gaussian and multiquadric functions small improvements in the accuracy can be achieved by very careful selection of this parameter. However this is not yet an obvious or an easy procedure.

References


Figure 1. A typical computational grid used in the dual-reciprocity solution is shown. In this case there are 146 external nodes, and 104 internal nodes arranged in a radial fashion (in a regular grid).

Figure 2. Variation of maximum errors of numerical solutions with Gaussian radial basis function parameter. The solid line corresponds to solution with modified RBF and Sobol distribution of internal collocation points. The dashed line corresponds to solution with regular RBF and regular internal point distribution.