Boundary element analysis of the interactions of cracks and holes in plane elastic bodies

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Abstract

By applying the new boundary integral formulation for two-dimensional elastic bodies containing cracks and holes proposed recently by Chau and Wang (1996), an effective boundary element method for calculating the interaction between cracks and holes is presented in this paper. In the crack tips, singular interpolation functions of order $r^{-1/2}$ are introduced to model crack tip singularity. The singular boundary integrals involved at the element level are integrated analytically. For each of the hole boundaries, an additional unknown constant is introduced such that displacement compatibility condition can be satisfied exactly by the complex boundary function $H(t)$. Another nice feature of the present formulation is that the stress intensity factors (both $K_I$ and $K_U$) in crack tips are expressed in terms of the nodal unknown of $H(t)$ exactly, and no extrapolation of data is required. To demonstrate the accuracy and efficiency of our boundary element method, an infinite elastic plane containing a circular hole and a kinked crack subject to far field uniaxial tension is considered; and their interaction is evaluated in terms of the change in stress intensity factors. When the hole is far from the crack or when the crack-hole interaction is negligible, our solutions agree well with the existing results for a kinked crack.

1 Introduction

Boundary element method has been developed into a powerful tool for solving various problems of fracture mechanics. Many effective formulations and techniques have been devised: the crack Green’s function method (e.g., Snyder and Cruse [1]); the displacement discontinuity method (e.g., Crouch [2]; the
subregions method (e.g., Blandford et al. [3]); the dual boundary element method (e.g., Portela et al. [4]); the dislocation density method (e.g., Weaver [5] and Wang [6]). In the most of these formulations for boundary element method for fracture mechanics, emphases have, however, been made to the numerical implementation of element discretization. Not much effort has been made in deriving a more compact and useful form of boundary integral equations which can be applied to solve analytic crack problems as well as to obtain numerical solution with much less computational effort. Along this line of analyses, Wang [6-8] has derived a simpler boundary integral formulation for three-dimensional elastic bodies containing coplanar cracks and obtained the exact solutions of boundary integral equations for an infinite elastic body with an internal or external circular crack under arbitrarily unsymmetric loadings. Some of Wang’s solutions have been tabulated into the stress intensity factor handbook by Murakami [9]. Recently, Chau and Wang [10] applied integration by parts to the Somigliana formula, then adopted the notations of the complex stress functions (e.g., see Muskhelishvili [11]), and finally obtained a new formulation for a system of boundary integral equations for plane elastic bodies containing arbitrary number of cracks and holes. In their formulation, the involved boundary integrals are interpreted in the sense of the principal value of the Cauchy type and a complex unknown function $H(t)$ is introduced, and it is basically a combination of the boundary traction and boundary displacement gradient. For some typical problems in plane elasticity and fracture mechanics, they have successfully obtained the exact solutions of the boundary integral equations and the results agree with the exact solutions obtained by other means.

For the more complex problems of interactions between cracks and holes of arbitrary sizes, an effective numerical scheme of boundary element method for the new boundary integral formulation [10] is presented in this study [12]. In the crack tips, singular interpolation functions of order $r^{-1/2}$ are introduced to model crack tip singularity, where $r$ is the distance measured from the crack tips. The singular boundary integrals involved at the element level are integrated analytically. Since the unknown $H(t)$ for our integral formulation is proportional to displacement gradient (not displacement), an additional unknown constant is introduced for each of the hole boundaries such that displacement compatibility condition can be satisfied exactly by the complex boundary function $H(t)$. Another nice feature of the present formulation is that the stress intensity factors (both $K_I$ and $K_{II}$) in crack tips are expressed in terms of the nodal unknown of $H(t)$ exactly, and no extrapolation of data is required. To demonstrate the accuracy and efficiency of our boundary element method, an infinite elastic plane containing a circular hole and a kinked crack subject to far field uniaxial tension is considered, and their interaction is evaluated in terms of the change in stress intensity factors. When the hole is far from the crack or when the crack-hole interaction is negligible, our solutions agree well with the existing results [13-16] for a kinked crack.
2 Boundary integral formulation

We consider the boundary value problems of multi-connected regions in plane elasticity and fracture mechanics. For infinite plane elastic bodies containing arbitrary number of cracks and holes (see Figure 1) without any body force, Chau and Wang [10] applied the technique of integration by parts to the Somigliana formula, then introduced a complex unknown function $H(t)$ relating to the boundary traction and boundary displacement gradient, and finally reduced the boundary integral formulations of stress and displacement to the following elegant expressions:

$\sigma_{11} + \sigma_{22} = 2[\Phi(z) + \overline{\Phi(z)}], \quad (z = x_1 + ix_2 \in \Omega \text{ and } i = \sqrt{-1}),$

$\sigma_{22} - \sigma_{11} + 2i\sigma_{12} = 2[\overline{z}\Phi'(z) + \Psi'(z)], \quad (\bar{z} = x_1 - ix_2),$

$2G(u_1 + iu_2) = \kappa\varphi(z) - z\varphi'(z) - \psi(z),$

where

$\Phi(z) = \frac{1}{4} (\sigma_{11}^\infty + \sigma_{22}^\infty) + \frac{1}{2\pi i} \int_{\Gamma} \frac{H(t)dt}{t-z},$

$\Psi(z) = \frac{1}{2} (\sigma_{22}^\infty - \sigma_{11}^\infty + 2i\sigma_{12}^\infty) - \frac{1}{2\pi i} \int_{\Gamma} \left[ \frac{H(t) - q(t)}{t-z} e^{-2i\alpha(t)} + \frac{tH(t)}{(t-z)^2} \right] dt,$

$\varphi(z) = -\frac{1}{4} (\sigma_{11}^\infty + \sigma_{22}^\infty)z - \frac{1}{2\pi i} \int_{\Gamma} H(t)\ln(t-z)dt + \gamma,$

$\psi(z) = \frac{1}{2\pi i} \int_{\Gamma} \left[ \frac{H(t) - q(t)}{t-z} e^{-2i\alpha(t)} \ln(t-z) - \frac{tH(t)}{t-z} \right] dt$

$+ \frac{1}{2} (\sigma_{22}^\infty - \sigma_{11}^\infty + 2i\sigma_{12}^\infty)z + \gamma',$

$H(t) = \frac{1}{\kappa + i} [q(t) + w(t)e^{-i\alpha(t)}] \quad \text{for } t = y_1 + iy_2 \in S + \Gamma,$

$q(t) = \sigma_n(y) + i\sigma_{n\bar{y}}(y) \quad \text{for } t \in S \text{ and } y = (y_1, y_2) \in S,$
In these formulae, $S$ denotes the union of the holes $S_1, S_2, \ldots, S_m$ and $\Gamma$ the union of the cracks $\Gamma_1, \Gamma_2, \ldots, \Gamma_n$. The shear modulus and Poisson’s ratio of elasticity are $G$ and $\nu$, and $k$ is equal to $3-4\nu$ for plane strain or $(3-\nu)/(1+\nu)$ for plane stress. The angle between the tangent at $t$ on $S+\Gamma$ and the global coordinate axis $\alpha_1$ is denoted by $\alpha(t)$, where o is the origin of the coordinate system. And $\sigma_{ij}$ and $u_i (i,j=1,2)$ are the components of stress and displacement in a Cartesian coordinate system $\alpha_1 \alpha_2$, respectively. The stresses $\sigma_{i}^\infty$ and $\sigma_{j}^\infty$ are given at infinity, as shown in Fig. 1. $\sigma_n$ and $\sigma_{ns}$ are the normal and shear stresses on the boundary. The superscripts “+” and “-” denote the upper and lower faces of cracks, respectively. $\gamma$ and $\gamma'$ are complex integral constants only related to rigid motion.

Letting the stresses (1) and (2) satisfy the traction boundary conditions on cracks and holes, the traction boundary integral equation at a smooth boundary point $t_0$ is derived as [10]

$$
\int_{S+\Gamma} \left\{ \frac{H(t)}{t-t_0} - \frac{H(t_0)}{t-t_0} e^{-2i\alpha(t_0)} + e^{-2i\alpha(t_0)} \left[ \frac{H(t) - q(t)}{t-t_0} + \frac{t-t_0}{(t-t_0)^2} \frac{H(t_0)e^{-2i\alpha(t)}}{t-t_0} \right] \right\} dt
$$

$$= \pi i [f(t_0) - g_1(t_0)], \quad (t_0 = x_{01} + ix_{02} \in S+\Gamma),
$$

where

$$g_1(t_0) = \sigma_{1}^\infty + \sigma_{2}^\infty + (\sigma_{1}^\infty - \sigma_{1}^\infty - 2i\sigma_{12}^\infty) e^{-2i\alpha(t_0)},
$$

$$f(t_0) = \sigma_n(x_0) + i\sigma_{ns}(\bar{x}_0) \quad \text{for } t_0 \in S,
$$

$$f(t_0) = [\sigma_n(x_0^+) + \sigma_n(x_0^-)] + i[\sigma_{ns}(x_0^+) + \sigma_{ns}(x_0^-)] \quad \text{for } t_0 \in \Gamma,
$$

$$e^{-i\alpha(t_0)} = -n_x(x_0) - in_1(x_0) \quad \text{for } x_0 = (x_{01}, x_{02}) \in S+\Gamma.
$$

The unknown function $H(t)$ in the boundary integral equation (13) must satisfy the following single-valuedness of displacement [10]:

$$\int_{S_k} H(t) dt = \frac{l}{k + l} \int_{S_k} q(t) dt \quad \text{for every hole } S_k (k=1,2,\ldots,m),
$$

$$\int_{\Gamma_j} H(t) dt = \frac{l}{k + l} \int_{\Gamma_j} q(t) dt \quad \text{for every embedded crack } \Gamma_j (j=1,2,\ldots,n).
$$

Once the complex unknown $H(t)$ is obtained from the boundary integral equation (13) and compatibility conditions (18-19), the stress intensity factors interested in fracture mechanics are calculated directly by [12]
where $a_j$ and $b_j$ are the complex coordinates of two tips of the $j$-th crack $\Gamma_j$.

3 Numerical implementation of boundary elements

For the general problems of plane elasticity and fracture mechanics, it is almost impossible to obtain the exact solutions of the boundary integral equation (13). Thus, in this section we discuss the numerical treatment of the boundary integral equation (i.e., boundary element method).

To get the numerical solution of the boundary integral equation (13) and the compatibility conditions (18) to (19), the cracks and holes are simply modelled by linear elements (see Figure 2), that is, $S + \Gamma = \sum L_e$.

![Figure 2: Crack and hole modelling with linear elements](image)

Each element $L_e$ is mapped onto the interval $-1 \leq \xi \leq 1$ or the spatial complex variable $t$ on the boundary is written as

$$t = t^*_e N_1(\xi) + t^*_e N_2(\xi), \quad (|\xi| \leq 1 \text{ and } t \in L_e);$$

where

$$N_1(\xi) = \frac{1}{2}(1 - \xi), \quad N_2(\xi) = \frac{1}{2}(1 + \xi);$$

$t^*_e$ and $t^*_e$ are complex coordinates of two nodes of the $e$-th element.

In general, the unknown boundary function $H(t)$ can also be interpolated linearly on both the holes and cracks in a form similar to that given in (21). However, when such interpolation is used along on hole's boundary, the number of unknowns will be less than the number of algebraic equations. To circumvent that difficulty, the following new interpolation function is proposed

$$H(t) = H^*_1 N_1(\xi) + H^*_2 N_2(\xi) + A_k, \quad (|\xi| \leq 1, \quad t \in L_e \subset S_k \text{ and } k = 1, 2, \ldots, m);$$

where $H^*_1 + A_k$ and $H^*_2 + A_k$ are the two nodal values of $H(t)$ on the ends of the element $L_e$ of the $k$-th hole; and $A_k$ is set equal for all elements on the same hole boundary but varies from one hole to the other.
When an element on cracks does not contain any crack tips, the interpolation function is taken as
\[ H(t) = H_t^e N_1(\xi) + H_t^e N_2(\xi) \quad (|\xi| \leq 1 \text{ and } t \in L_e \subset \Gamma) \] (24)

Otherwise, if the local node 1 ($\xi = -1$) of an element is at a crack tip, the unknown $H(t)$ is approximated by
\[ H(t) = \frac{H_t^e}{\sqrt{N_1(\xi)}} + (H_t^e - H_t^e) N_2(\xi) \quad (|\xi| \leq 1 \text{ and } t \in L_e \subset \Gamma) \] (25)

Similarly, in the case of the element node 2 ($\xi = 1$) being a crack tip, $H(t)$ is represented as
\[ H(t) = (H_t^e - H_t^e) N_1(\xi) + \frac{H_t^e}{\sqrt{N_1(\xi)}} \quad (|\xi| \leq 1 \text{ and } t \in L_e \subset \Gamma) \] (26)

where the shape functions $N_1(\xi)$ and $N_2(\xi)$ in (23) to (26) are again given by (22).

Substituting the interpolation functions (23) to (26) into the boundary integral equation (13) and compatibility conditions (18) to (19) and taking a complex variable $t_0$ in (13) in turn as the midpoint of each element, the solution of the boundary integral equation is reduced to solve a closed system of linear algebraic equations. After obtaining the solution of these equations, the stress intensity factors are determined directly from
\[ K_1(a_j) - iK_{II}(a_j) = -\sqrt{2\pi t^e_2 - t^e_1} \cdot iH_t^e \quad \text{for} \quad t^e_1 = a_j, \]
\[ K_1(b_j) - iK_{II}(b_j) = \sqrt{2\pi t^e_2 - t^e_1} \cdot iH_t^e \quad \text{for} \quad t^e_2 = b_j, \]

(j=1,2,...,n, $i = \sqrt{-1}$) (27)

which is obtained by inserting (25) and (26) into (20), where $a_j$ and $b_j$ are the complex coordinates of two tips of the $j$-th crack $\Gamma_j$.

4 Evaluation of singular integrals

In the analysis of boundary elements, the accuracy and efficiency depends mainly on how we evaluate the singular integrals involved. For the ordinary boundary integral equations, it can be solved by introducing the rigid body translation. But that technique is unsuitable for the present boundary integral equations because displacement gradient instead of displacement is used in our formulation. Hence the evaluation of the singular element integrals is investigated and discussed below.

As mentioned in the previous section, the boundary integral in (13) is equal to the sum of element integrals, that is, \[ \int_{s\cap\Gamma} \cdots dt = \sum_{s\cap\Gamma} \int_{t} \cdots dt, \] in which the ordinary element integrals can be evaluated by the Gauss-Legendre integration formula. When the source point $t_0$ belongs to $L_e$, the element integral is singular and
Substituting the interpolation functions (23) to (26) into (28) results in the exact singular element integrals respectively as follows

\[ \int_{-1}^{1} \frac{2H(t(\xi)) - q(t(\xi))}{\xi - \xi_0} \, d\xi, \quad (t_0 = t^* N_1(\xi_0) + t^*_2 N_2(\xi_0) \in L_e and |\xi_0| < 1), \]  

(28)

where

\[ L_e = \left\{ t | t = t^*_1 N_1(\xi) + t^*_2 N_2(\xi) \text{ and } |\xi| \leq 1 \right\}. \]  

(29)

Substituting the interpolation functions (23) to (26) into (28) results in the exact singular element integrals respectively as follows

\[ \int_{-1}^{1} \frac{q(t(\xi))}{\xi - \xi_0} \, d\xi = q(t(\xi_0)) \ln \left| \frac{1 - \xi_0}{1 + \xi_0} \right| + (q^*_1 - q^*_1); \]

\[ \int_{-1}^{1} \frac{H(t(\xi))}{\xi - \xi_0} \, d\xi = H(t(\xi_0)) \ln \left| \frac{1 - \xi_0}{1 + \xi_0} \right| + (H^*_2 - H^*_1), \quad \text{for ordinary elements;} \]

\[ \int_{-1}^{1} \frac{H(t(\xi))}{\xi - \xi_0} \, d\xi = \frac{H^*_1}{\sqrt{N_2(\xi_0)}} \ln \left| \frac{1 - \sqrt{N_2(\xi_0)}}{1 + \sqrt{N_2(\xi_0)}} \right| + (H^*_2 - H^*_1) N_2(\xi_0) \ln \left| \frac{1 - \xi_0}{1 + \xi_0} \right| \]

\[ + (H^*_2 - H^*_1), \quad \text{for initial crack tip elements;} \]

\[ \int_{-1}^{1} \frac{H(t(\xi))}{\xi - \xi_0} \, d\xi = - \frac{H^*_2}{\sqrt{N_1(\xi_0)}} \ln \left| \frac{1 - \sqrt{N_1(\xi_0)}}{1 + \sqrt{N_1(\xi_0)}} \right| + (H^*_1 - H^*_2) N_1(\xi_0) \ln \left| \frac{1 - \xi_0}{1 + \xi_0} \right| \]

\[ + (H^*_2 - H^*_1), \quad \text{for end crack tip elements;} \]  

(30)

If \( t_0 \) is taken as the midpoint of a singular element, \( \xi_0 \) in (30) is equal to zero. Since all singular integrands are evaluated exactly, the proposed numerical scheme is extremely efficient and accurate.

5 Effect of a circular hole on a crack in an infinite body

To demonstrate the validity and applicability of the present boundary element technique, the effect of a circular hole on a crack in an infinite elastic body subjected to remote uniform tension or compression is considered. In the following examples, the crack faces and hole boundaries are taken as traction free, and all calculations are performed on a PC/586 computer.

The normalized stress intensity factor \( F_i = K_i / (\sigma \sqrt{a}) \) of the problem of an infinite elastic plane containing a circular hole and a radial crack under the uniaxial uniform tension is given in Figure 3, which agrees well with the published results [17]. Figure 4 shows the effect of a circular hole on a crack under remote uniaxial compression, which agrees with the prediction from the exact solution [18] for a circular hole because the tensile stress appears only in the interval \([R, \sqrt{3}R]\) on the crack line. For an infinite plate containing
a kinked crack and a circular hole under remote uniaxial tension (see Figure 5), the normalized stress intensity factors \( F_1 = K_1 / (\sigma \sqrt{\pi a}) \) and \( F_2 = K_2 / (\sigma \sqrt{\pi a}) \) are given in Figures 6-7 and Table 1. Figures 6-7 show the effects of a circular hole on a kinked crack. When the hole is far from the crack, the crack-hole interaction is negligible. And the present BEM solutions listed in Table 1 agree well with the existing results [13-16] for a kinked crack.

![Figure 3: Tension of infinite plate with a crack and a circular hole](image)

![Figure 4: Compression of infinite plate with a crack and a circular hole](image)

![Figure 5: Tension of infinite plate with a kinked crack and a circular hole](image)

![Figure 6: Variation of \( F_1 = K_1 / (\sigma \sqrt{\pi a}) \) with \( R/a \)](image)

![Figure 7: Variation of \( F_2 = K_2 / (\sigma \sqrt{\pi a}) \) with \( R/a \)](image)
Table 1: Normalized stress intensity factors for a kinked crack far from a circular hole in an infinite plate under uniaxial tension ($b/a=0.2$, $R/a=0.2$, $h/a=0.1$, $d/a=10$ and $\theta=-45$ degrees)

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<th>$K_{II}(A)/\sigma \sqrt{\pi a}$</th>
<th>$K_I(B)/\sigma \sqrt{\pi a}$</th>
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6 Conclusions

An effective boundary element method has been established to solve a problem of plane elasticity and fracture mechanics with multiple holes and cracks. The new linear interpolation functions for $H(t)$ on hole’s elements and singular interpolation functions of order $r^{-1/2}$ on crack tip elements are proposed. The singular integrals on elements are treated analytically. The examples of an infinite elastic plate containing a circular hole and a crack (straight or kinked) under remote uniaxial tension or compression are given, which show both the effect of a circular hole on a crack and the advantages of present method in dealing with different numbers, distributions, orientations and shapes of holes and cracks.

Acknowledgements

The work was supported by RGC Grant No.354/052 to The Hong Kong Polytechnic University through KTC and by the Natural Science Foundation of the Gansu Province under Contract No.ZQ-95-005 to Lanzhou University through YBW.

key words: crack and hole; boundary element; singular interpolation function; exact singular integral; stress intensity factor

References


