Boundary element analysis of 3-D problems in coupled thermoelasticity

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This paper deals with a new boundary element method for analysis of the quasi-static problems in coupled thermoelasticity. Through some mathematical manipulation of the Navier equation in elasticity, the heat conduction equation is transformed into a simpler form, similar to the uncoupled-type equation with the modified thermal conductivity which shows the coupling effects. This procedure enables us to treat the coupled thermoelastic problems as an uncoupled one. A few examples are computed by the proposed BEM, and the results obtained are compared with the analytical ones available in the literature, whereby the accuracy and versatility of the proposed method are demonstrated.

Introduction

Computational mechanics has been so well developed that the computer analysis software can be successfully applied to many problems in science and engineering. In recent years, considerable attention has been paid to the numerical analysis of coupled thermoelasticity, which has a wide range of application in engineering and technology, for example, in precise design of electromagnetic devices used in high-technologies.

Coupled thermoelasticity encompasses the phenomena that describe the elastic and thermal behavior of solids and their interactions under mechanical and thermal loadings. Many attempts have been directed toward the solution of the uncoupled thermoelasticity problems in steady or transient heat conduction states, but a smaller number of investigations have been done successfully for coupled thermoelasticity problems. Analytic solutions for some of dynamic problems in
coupled thermoelasticity were obtained by Danilovskaya [1-2], and Strengberg and Chakravorty [3]. For some of the quasi-static problems in coupled thermoelasticity Boley and Tolins [4] obtained analytic solutions, while Nickell and Sackman [5] presented approximate solutions. The boundary element method, recognized in recent years as a powerful tool for numerical analysis, was applied by Rizzo and Shippy [6] to the solution of steady-state thermoelastic problems. Concerning the BEM solution of transient, uncoupled problems, we can refer to Tanaka et al. [7], and Sladek and Sladek [8]. To authors' best knowledge, the boundary element methods for the coupled problems in thermoelasticity with numerical computation of some examples have been reported by Suh and Tosaka [9], Ishiguro and Tanaka [10], and Dargush and Banerjee [11]. The former two papers use a matrix calculation method, originally presented by Hörmander [12] and used by Kupradze [13] for some of coupled problems in thermoelasticity, to derive the fundamental solution tensor from the adjoint differential equations of the coupled thermoelastic problem. Suh and Tosaka use the Laplace transform, while Ishiguro and Tanaka propose a BEM based on the time-stepping approximation of time derivatives. Dargush and Banerjee presented a BEM solution implementing a reciprocal theorem for quasi-static poroelasticity or thermoelasticity problems.

As has been well known, there are the following kinds of BEM available for time-dependent problems:

1. Laplace or Fourier transform method
2. Direct method
3. Time-stepping method
4. Multiple reciprocity method
5. Boundary-domain element method

In this paper, a direct method of the boundary element method is developed for the quasi-static, problems in coupled thermoelasticity. In the present approach the heat conduction equation is transformed into a simpler form through some mathematical manipulation of the Navier equation in elasticity, in order to reduce the complexity of the boundary integral formulation of the problem and also to save computing time as well as to reduce errors due to this complexity. The new heat conduction equation has the form of an uncoupled-type equation with thermal conductivity which shows the coupling effects. Therefore, we may formulate the boundary integral equation of the problem in the usual manner of the BEM. The formulation is implemented by the standard BEM using constant elements, and the potential usefulness and accuracy of the proposed method are demonstrated through numerical computation for some of the 3-D problems in coupled thermoelasticity and comparison with other results available in the literature.
It is noted that Einstein's indicial summation convention is implied throughout this paper, and also the superimposed dot denotes the derivative with respect to time.

**Basic Field Equations**

Let $\Omega$ be the domain occupied by an isotropic, homogeneous elastic medium bounded by the boundary $\Gamma$ in the three dimensional Euclidean space. The governing equations of linear, quasi-static, coupled thermoelasticity problems can be expressed by the following system of differential equations:

$$\mu u_{i,jj} + (\lambda + \mu)u_{j,ij} - \gamma \theta, i + \rho f_i = 0$$  \hspace{1cm} (1)

$$\theta_{,jj} - \frac{1}{k} \theta - \eta u_{j,j} + \frac{1}{k} Q = 0$$  \hspace{1cm} (2)

where $u_i, \theta, f_i$ and $Q$ are displacement, temperature, body force and heat source at an arbitrary point of the medium. The Lamé constants are denoted by $\lambda$ and $\mu$, whereas density and thermal diffusivity by $\rho$ and $k$, respectively. The thermoelasticity constants are $\gamma$ and $\eta$, which can be defined as follows:

$$\gamma = (3\lambda + 2\mu)\alpha, \quad \eta = \gamma T_0 / k C_v$$  \hspace{1cm} (3)

where $\alpha$ and $C_v$ are the coefficient of linear heat expansion, and the specific heat at a uniform reference temperature $T_0$, respectively. The constitutive equation is given by the Duhamel-Neumann law as follows:

$$\sigma_{ij} = 2\mu \epsilon_{ij} + (\lambda \theta - \gamma \theta) \delta_{ij}$$  \hspace{1cm} (4)

where $\sigma_{ij}, \epsilon_{ij}$ and $\theta$ are stress and strain components, and volume expansion, respectively. The strain components and volume expansion can be related to the displacements as

$$\epsilon_{ij} = \frac{1}{2} (u_{i,j} + u_{j,i}), \quad \theta = u_{i,i}$$  \hspace{1cm} (5)

By taking divergence of equation (1), we can obtain the following relation between the volume expansion and the temperature:

$$\nabla^2 [(\lambda + 2\mu)e - \gamma \theta] = -\rho f_{i,i}$$  \hspace{1cm} (6)

An important relationship can be derived by integrating equation (6), which is expressed by

$$(\lambda + 2\mu)e - \gamma \theta = F(x, y, z, \text{cons.})$$  \hspace{1cm} (7)
where $F$ is a function of coordinates or a constant. This is obviously independent of time, and hence derivating it with respect to time, we can obtain

$$ (\lambda + 2\mu)\dot{e} = \gamma \dot{\theta} \quad (8) $$

Substituting (8) into (2), we obtain

$$ \theta_{,ij} - \frac{1}{k} \left( 1 + \frac{\eta \gamma k}{\lambda + 2\mu} \right) \dot{\theta} + \frac{1}{k} Q = 0 \quad (9) $$

where $\eta \gamma k / (\lambda + 2\mu)$, which is non-dimensional and will later be denoted by $CE$, shows the coupling effects. Let us denote by $k_c$ a new coupled heat conduction coefficient:

$$ k_c = k \left( \frac{\lambda + 2\mu}{\lambda_a + 2\mu} \right) \quad (10) $$

where

$$ \lambda_a = \lambda + \eta \gamma k \quad (11) $$

Then, equation (9) can be rewritten as

$$ \theta_{,ij} - \frac{1}{k_c} \dot{\theta} + \frac{1}{k} Q = 0 \quad (12) $$

Equations (1) and (12) are the basic field equations used in this paper.

**Integral Equation Formulation**

Let us consider weighted residual statements of the governing equations (1) and (12), in which the weight functions are the fundamental solutions. From equation (1), we obtain

$$ \int_{\Omega} (\mu u_{i,jj} + (\lambda + \mu)u_{j,j} - \gamma \theta_{,j} + \rho f_j)U^*_ik d\Omega = 0 \quad (13) $$

and from the coupled conduction heat equation (12)

$$ \int_{\partial \Omega} \left( \theta_{,ij} - \frac{1}{k_c} \dot{\theta} + \frac{1}{k} Q \right) \theta^* d\Omega = 0 \quad (14) $$

Integrating them over the whole spatial domain $\Omega$, and then applying the divergence theorem as well as integration by parts w.r.t. time, we can derive the following boundary integral equations:

$$ c \delta_{km} \dot{u}_m(p) = \int_{\Gamma} t_i(p') U^*_ik(p,p') d\Gamma(p') - \int_{\Gamma} u_i(p') T^*_ik(p,p') d\Gamma(p') $$

$$ + \gamma \int_{\Omega} U^*_{ik,j}(p,p'') \theta(p'') d\Omega(p'') + \int_{\Omega} \rho f_i U^*_ik(p,p'') d\Omega(p'') \quad (15) $$
and
\[
c(\mathbf{p}) = \int_0^t \int_{\Gamma} \hat{\theta}(\mathbf{p}',\mathbf{p}) q(\mathbf{p},\mathbf{p}') d\Gamma(\mathbf{p}') d\tau - \int_0^t \int_{\Gamma} q(\mathbf{p}') \theta(\mathbf{p},\mathbf{p}') d\Gamma(\mathbf{p}') d\tau \\
+ \int_{\Omega} \theta \theta(\mathbf{p},\mathbf{p}'') d\Omega(\mathbf{p}'') + \int_0^t \int_{\Omega} \frac{k}{k} Q \theta(\mathbf{p},\mathbf{p}'') d\Omega(\mathbf{p}'') d\tau
\]

(16)

where \( \mathbf{p} \) is the source point while \( \mathbf{p}' \) and \( \mathbf{p}'' \) are field points on the boundary and domain respectively; \( c \) is the coefficient which is determined from the boundary geometrical shape at the source point; \( T^*(\mathbf{p},\mathbf{p}') \) and \( q^*(\mathbf{p},\mathbf{p}') \) are traction and heat flux of fundamental solutions; and \( t_i \) and \( q \) are traction vector and heat flux, which can be related to stress and temperature gradient respectively as follows:
\[
t_i = \sigma_{ij} n_j, \quad q = -\theta_j n_j
\]

(17)

where \( n_j \) is the component of a unit outward normal vector. The fundamental solutions for a three dimensional isotropic medium are given by [14]
\[
U^*_{ik} = \frac{1}{16\pi \mu (1-\nu)} \frac{1}{r} \left[ (3-4\nu)\delta_{ik} + r_i r_k \right]
\]

(18)

\[
\theta^* = \frac{\exp \left[ -\frac{r^2}{4k_c(t-\tau)} \right]}{\left[ 4\pi k_c(t-\tau) \right]^{3/2}}
\]

(19)

\[
T^*_{ik} = \frac{-1}{8\pi (1-\nu)} \frac{1}{r^2} \left[ (1-2\nu)(r_i n_k - r_k n_i + r_m n_m \delta_{ik}) + 3 r_i r_k r_m n_m \right]
\]

(20)

\[
q^* = \theta^* r_m n_m \left[ \frac{r}{2k_c(t-\tau)} \right]
\]

(21)

where \( r \) is the distance between two points \( \mathbf{p} \) and \( \mathbf{p}' \) (or \( \mathbf{p}'' \)) as \( r = \| \mathbf{p} - \mathbf{p}' \| \).

We can calculate unknown displacements and tractions on the boundary by solving the BIE (15) under the given boundary conditions, after having known from the BIE (16) the temperature distribution over the whole domain under consideration.

**Numerical Implementation**

In order to solve boundary integral equations (15) and (16) numerically, it is necessary to discretize the boundary \( \Gamma \) and domain \( \Omega \) into a number of boundary elements and cells, respectively, whereas the time axis is divided into small intervals, which can be of equal size \( \Delta \tau \). In the simplest, possible implementation of the formulation, we may utilize the assumption that all the
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elastic and thermal variables are piecewisely constant on the boundary at each time step. Such BEM implementation has been successfully applied to many of practical problems [7, 9, 14].

In the BIE(16), time is marched from \( \tau = 0 \) to any specified time \( (t = t) \), in the step-by-step manner by using the constant time increment \( \Delta \tau \). For this purpose, we may use the following two ways:

(1) At any time interval, we solve the following boundary integral equation:

\[
c\theta(p) = \int_0^{\Delta \tau} \int_{\Gamma} \theta(p')q^*(p,p')d\Gamma(p')d\tau - \int_0^{\Delta \tau} \int_{\Omega} q(p')\theta^*(p,p')d\Gamma(p')d\tau + \int_{\Omega} \theta_0 \theta^*(p,p'')d\Omega(p'') + \int_0^{\Delta \tau} \int_{\Omega} \frac{k}{k} Q \theta^*(p,p'')d\Omega(p'')d\tau \quad (22)
\]

The data used, are obtained at the previous time step as the pseudo initial condition. It should be noted that, at \( \tau = 0 \), the initial condition is specified.

(2) After discretizing the time axis into constant elements, we can write the BIE(16) at the \( N \)th time step as follows:

\[
c\theta(p) = \int_{t_{N-1}}^{t_N} \int_{\Gamma} \theta(p')q^*(p,p')d\Gamma(p')d\tau - \int_{t_{N-1}}^{t_N} \int_{\Omega} q(p')\theta^*(p,p')d\Gamma(p')d\tau + \sum_{i=1}^{i-1} \int_{t_{i-1}}^{t_i} \int_{\Gamma} \theta(p')q^*(p,p')d\Gamma(p')d\tau - \sum_{i=1}^{N-1} \int_{t_{i-1}}^{t_i} \int_{\Omega} q(p')\theta^*(p,p')d\Gamma(p')d\tau + \int_{\Omega} \theta_0 \theta^*(p,p'')d\Omega(p'') + \int_0^{\Delta \tau} \int_{\Omega} \frac{k}{k} Q \theta^*(p,p'')d\Omega(p'')d\tau \quad (23)
\]

in which the third and fourth integrals on the right-hand side involve the known information from the solution of the previous \( (N-1) \) time steps. The time integration in equation (22) or (23) can be carried out analytically as follows [14]:

\[
\int_{t_{k-1}}^{t_k} \theta^*(p,p')d\tau = \frac{1}{4\pi\sqrt{r}r'} \left[ \Gamma(1/2,Z_{k-1}) - \Gamma(1/2,Z_k) \right]
\]

\[
\int_{t_{k-1}}^{t_k} q^*(p,p')d\tau = \frac{r_n}{2\pi\sqrt{r}r'} \left[ \Gamma(3/2,Z_{k-1}) - \Gamma(3/2,Z_k) \right] \quad (24)
\]

where \( Z_k = r^2/4k_c(t - \tau_k) \) and \( \Gamma \) is the Gamma function, defined by

\[
\Gamma(m,Z) = \int_Z^{\infty} e^{-\lambda} \lambda^{m-1}d\lambda \quad (25)
\]

In this paper, the solutions of the above mentioned time marching procedures, based on equations (22) and (23), are called BEM1 and BEM2, respectively.

Discretizing equations (15) and (16) by means of the boundary element method, and then imposing the boundary conditions, we can obtain the following system of linear equations:
\[ [A][X] = [B][Y] \]  

(26)

where \{X\} denotes the unknown vector of nodal values on the boundary; \{Y\} is the known one; \([A]\) and \([B]\) are coefficient matrices evaluated from boundary integrals.

As computational examples, the first and second coupled problems of Danilovskaya are chosen to demonstrate the validity and versatility of the proposed method. The Danilovskaya problems are essentially one-dimensional ones, and in the numerical comparison, the analytical solutions shown in this paper were obtained through the numerical inversion of the exact solutions in Laplace transformed domain.

The first Danilovskaya problem is concerned with a linear elastic half-space \((x>0)\), subjected to a uniform sudden temperature change on its bounding plane \((x=0)\), under the assumption of traction free boundary conditions on the plane \(x=0\). The medium is assumed to be mechanically constrained and thermally insulated in the \(y\) and \(z\) axes, so that displacements and temperature can be expressed as

\[
    u = u_x(x,t) \quad , \quad u_y = u_z = 0 \quad , \quad \theta = \theta(x,t)  
\]

(27)

The boundary conditions of the bounded plane are given by

\[
    t_i(x,t) = 0 \quad , \quad \theta(x,t) = T_0 H(t) \quad , \quad \text{on} \quad x = 0  
\]

(28)

\(H(t)\) denotes the Heaviside step function; \(T_0\) is temperature of bounding plane at \(t \geq 0\).
In the second problem of Danilovskaya, the bounding plane (x=0) will be assumed to be exposed to a high ambient temperature through a boundary layer of finite thermal conductance. In this case, the temperature condition (28) is replaced by

\[ q(x,t) = h(T(x,t) - T_\infty) \]  

(29)

where \( h \) is the convection heat transfer coefficient; \( T \) and \( T_\infty \) are the temperatures of bounding plane and ambience, respectively. The geometry of the problems in the half-space with the reference length of \( x=6[\text{mm}] \) is depicted in Fig. 1, and its three dimensional model is discretized to 77 boundary elements and 22 cells as shown in Figure 2.

The other mechanical and thermal conditions are the same as the infinite layer. For the sake of brevity, we assume homogeneous initial conditions, no body force and no internal heat sources. The material and thermal properties of the elastic body are assumed as follows:

- Coefficient of thermal expansion : \( \alpha = 1.1 \times 10^{-5} [\text{°C}] \);
- Poisson’s ratio : \( \nu = 0.3 \);
- Young’s modulus : \( E = 210 [\text{GPa}] \);
- Thermal diffusivity : \( k = 16 [\text{mm}^2 / \text{s}] \)  

(30)

Figure 2: Three dimensional model and boundary element mesh
Figure 3: Results on the first problem of Danilovskaya (coupled as $CE=0.5$)
Figure 4: Results on the second problem of Danilovskaya (coupled as $CE=0.5$, $h=0.1$)
Numerical integration is carried out by the standard Gauss-Quadrature method with four or six points, except for the singular integrals that should be evaluated analytically.

The results of the first problem of Danilovskaya with two different time marching procedures are presented in Figure 3. Figures 3 (a) and (b) show the time variations of displacement and temperature on a particular location, $x=1$ [mm] far from the mid-point of the bounding plane, when the coupling effect parameter is assumed as $C_E=0.5$. In these calculations, the temperature $T_0$ of bounding plane in the boundary condition (28) is supposed to be 100 [$^\circ$C].

Figures 4 (a) and (b) show time variations of displacement and temperature at the center point on plane $x=1$ for the second problem of Danilovskaya with the convection heat transfer coefficient $h=0.1$ [mm/s] and the ambient temperature of 100 [$^\circ$C].

The present results obtained by the boundary element method, especially BEM2, are in very good agreement with the exact solutions shown by solid lines, although a coarse discretization was used. It should be noted, however, that BEM2 consumes more computing time than BEM1.

**Conclusion**

In this paper, a boundary element method based on the so-called direct approach has been developed for obtaining the solutions of three dimensional quasi-static, coupled thermoelasticity problems. As numerical examples, the first and second problems of Danilovskaya have been tested by the computer program developed in this study. Simplicity of the formulations enables us to obtain in an efficient manner the computational results within smaller computing time rather than the existing methods of solution. This procedure can be easily applied to the corresponding 2-D problems.

The present formulation is concerned only with the quasi-static problem in coupled thermoelasticity. In the context of thermoelasticity, however, we may conclude that the proposed quasi-static formulas are applicable without major loss of generality to most of the practical thermoelastic problems in engineering.
References