Fundamental solutions of circular inclined rigid punch on a half plane with an oblique edge crack
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Abstract

A circular rigid punch is located on a half plane with an oblique edge crack. The punch is acted by a vertical load at its center, and frictional force is assumed to exist on the contact region and applies in the horizontal direction. A pair of concentrated forces is assumed to act at arbitrary point in the half plane. Owing to the existence of the crack, the frictional force and the concentrated forces or point dislocations, the punch is usually inclined and the inclined angle is decided by the condition that the resultant moment about the center on the contact region must be vanished. The punch is supposed to contact with the half plane with two sharp corners at both ends, therefore the length of the contact region has been given. To find the analytical solution of the problem, a rational mapping function which maps the half plane with an oblique edge crack into a unit circle is used. According to the loading and displacement conditions, the problem can be transformed into the Riemann-Hilbert problem. To solve the problem, the complex stress functions are divided into two parts; one is the principle part, which corresponds to the fundamental solution of the half plane with an oblique edge crack; the other is the holomorphic part of the problem, which can be derived explicitly by solving Riemann-Hilbert equation.

1 Introduction

It is well known that the fundamental solutions of an infinite plane and a semi-
infinite plane subjected to concentrated forces or point dislocations at arbitrary points play important roles in the analysis of various problems in engineering, especially in the application of Boundary Element Method [1,3].

A rigid flat or circular punch problems on a half plane with a crack have been studied in the previous papers [5,6,8] using complex stress functions. In the present paper, the fundamental solution of the half plane with an edge crack [2] is used to derive the fundamental solution of the punch problem on the cracked half plane subjected to concentrated forces. The half plane with an edge crack is first mapped into a unit circle by a rational mapping function so that the forward derivation can be performed on the mapped plane in an analytical way. The fundamental solution of the half plane with an edge crack is derived by making use of the regularity of the complex stress functions of the half plane. According to the loading and displacement conditions, the punch problem can be transformed into the Riemann-Hilbert problem. To solve the R-H equation, the complex stress functions for the whole problem are divided into two parts, one is the principle part, which is corresponding to the solution of the half plane with an edge crack acted by concentrated forces, which has been presented by another paper of ours [2]; the other is the holomorphic part of the problem. By substituting the first part into the R-H equation, and introducing a Plemelj function, the solution of the second part can be obtained explicitly. Since the punch is usually inclined owing to the existence of the crack, the frictional force and the concentrated forces or point dislocations, the influence of the inclined angle of the punch is separated from the whole problem, i.e., two problems; one is that the circular punch does not incline and the other is that the flat punch is inclined. The inclined angle of the punch can be decided by the condition that the resultant moment on the contact region must be zero.

2 The mapping function

The following rational mapping function is used to map the half plane with an oblique crack into a unit circle [4]

\[ z = \omega(\zeta) = \frac{E_0}{1 - \zeta} + \sum_{\xi} \frac{E_{\xi}}{\xi - \zeta} + E_x \]  \hspace{1cm} (1)
where $E_o$, $E_k$ and $\zeta_k (|\zeta_k| > 1)$ are known constants, $E_c$ is related to the location of the origin of the coordinates, and $N = 24$ is used in this paper. The values of the coefficients of (1) have been shown in [9].

3 The loading and displacement conditions for the problem

The problem is shown in Fig.1, in which the circular rigid punch is acted by a vertical load $P$ at the center of the punch. Coulomb's frictional force is supposed to exist on the contact region. An oblique edge crack with an angle $\gamma$ ($0 < \gamma < 180^\circ$) is located at or away from the right end of the punch. The half plane is assumed to be subjected to concentrated forces at an arbitrary point in the body of the half plane.

It is supposed that the half plane acted by a pair of concentrated forces $q_x, q_y$ at arbitrary point $z_0$ and another pair of concentrated forces $-q_x, -q_y$ at another point $z_m$. The two pairs of concentrated forces are in self-equilibrium. When $z_m$ is infinity, the functions to be determined is obtained by letting $z_m \to \infty$, i.e. $\zeta_m \to 1$, which corresponds to infinity.

The punch is usually inclined owing to the existence of the crack, the frictional force and the concentrated forces or point dislocations. The inclined angle of the punch is taken as positive value when the punch is inclined in clockwise direction, and it can be decided by the condition that the resultant...
moment about the center (origin of the coordinate system) of the contact region must be zero in the natural state. The problem is separated into two parts A and B according to the characteristics of the loading and displacement conditions of the problem. Part A is due to the circular punch acted by an eccentric load and subjected to two pairs of concentrated forces in the half plane to keep the punch vertical by the moment on the contact region; part B is due to a flat-ended punch inclined with an angle \( \epsilon \) by the moment on the contact region without any loads.

The loading and displacement conditions for each part can be presented as follows, respectively:

For part A,

\begin{align*}
  p_x &= p_y = 0 & \text{on } L = L_1 + L_2 \\
  p_x &= \mu p_y, \int p_y \, ds = P & \text{on } M \\
  V_A &= x^2 / 2R & \text{on } M
\end{align*}

The condition related to the concentrated forces is expressed as

\begin{align*}
  Q(x, y) = (q_x + iq_y) \delta(z, z_0) - (q_x + iq_y) \delta(z, z_m)
\end{align*}

for part B,

\begin{align*}
  p_x &= p_y = 0 & \text{on } L = L_1 + L_2 \\
  p_x &= \mu p_y, \int p_y \, ds = 0 & \text{on } M \\
  V_B &= -\epsilon x & \text{on } M
\end{align*}

where \( L_1 = ABCD' D, L_2 = EA, M = DE \) in Fig. 1; \( \mu \) represents the Coulomb's frictional coefficient on M; \( p_x \) and \( p_y \) represent the components of traction in x and y directions on the surface of the half plane; \( Q(x, y) \) represents the concentrated forces in the half plane; \( \delta(z, z_0) = 1 \) when \( z = z_0 \) and 0 when \( z \neq z_0 \), so does \( \delta(z, z_m) \). \( V_A \) and \( V_B \) represent the displacements on the contact region for parts A and B, respectively; \( R \) represents the radius of curvature of the punch, and \( \epsilon \) is the inclined angle of the punch.

4 The fundamental solution of the problem

According to the above loading and displacement conditions, each part can be transformed into the Riemann-Hilbert problem as follows \([5,8]\):

\begin{align*}
  \phi_j^+(\sigma) - \phi_j^-(\sigma) &= f_{ij} & \text{on } L = L_1 + L_2 \\
  \phi_j'(\sigma) + \frac{1}{g} \phi_j(\sigma) &= f_{kj} & \text{on } M
\end{align*}

where

\begin{align*}
  f_{ij} &= i \int (p_{xj} + ip_{yj}) \, ds
\end{align*}


\[ f_{A0} = \frac{4(1 - i\mu)GiV_j + (1 + i\mu)(1 + \kappa)R_j(\sigma)}{\kappa + 1 - i\mu(\kappa - 1)} \]  

(6b)

\[ R_j(\zeta) = \phi_j(\zeta) + \frac{1 - i\mu}{1 + i\mu} \phi_j\left(\frac{1}{\zeta}\right) \]  

(6c)

\[ \frac{1}{g} = \frac{(\kappa + 1) + i\mu(\kappa - 1)}{\kappa + 1 - i\mu(\kappa - 1)} \]  

(6d)

and \( j = A, B \) represent parts A and B, respectively, \( V_A = x^2 / 2R \) for part A and \( V_B = -ex \) for part B. \( R_j(\zeta) \) is a function to be determined so as to satisfy (6c), and \( G \) is the shear modulus of the half plane. \( \kappa = 3 - 4\nu \) for plane strain state and \( (3 - \nu)/(1 + \nu) \) for plane stress state, respectively, and \( \nu \) represents the Poisson’s ratio of the half plane.

### 4.1 The solution of part A

The complex stress functions to be obtained are represented by two terms:

\[ \phi_A(\zeta) = \phi_{A1}(\zeta) + \phi_{A2}(\zeta) \]  

(7a)

\[ \psi_A(\zeta) = \psi_{A1}(\zeta) + \psi_{A2}(\zeta) \]  

(7b)

where \( \phi_{A1}(\zeta) \) and \( \psi_{A1}(\zeta) \) are the complex stress functions of the half plane acted by the two pairs of concentrated forces[2]. \( \phi_{A2}(\zeta) \) and \( \psi_{A2}(\zeta) \) are the holomorphic parts of \( \phi_A(\zeta) \) and \( \psi_A(\zeta) \), respectively.

Substituting (7a) into (5), it is obtained that

\[ \phi_{A2}(\sigma) - \phi_{A2}(\sigma) = f_{LA2}(\sigma) \]  

(8a)

\[ \phi_{A2}(\sigma) + \frac{1}{g} \phi_{A2}(\sigma) = f_{MA2}(\sigma) + C \left[ \frac{\phi_{A2}(\sigma)}{\phi_{A1}(\sigma)} - \phi_{A1}(\sigma) \right] \]  

(8b)

where

\[ f_{LA2}(\sigma) = \begin{cases} 0 & \text{on } L_1 \\ P(1 - i\mu) & \text{on } L_2 \end{cases} \]  

(9a)

\[ f_{MA2}(\sigma) = \frac{4(1 - i\mu)GiV_A + (1 + i\mu)(1 + \kappa)R_A(\sigma)}{\kappa + 1 - i\mu(\kappa - 1)} \]  

(9b)

\[ R_A(\zeta) = \phi_{A2}(\zeta) + \frac{1 - i\mu}{1 + i\mu} \phi_{A2}\left(\frac{1}{\zeta}\right) \]  

(9c)

\[ C = \frac{(1 - i\mu)(\kappa + 1)}{(\kappa + 1) - i\mu(\kappa - 1)} \]  

(9d)

By introducing the Plemelj function \( \chi(\zeta) \), the general solution of (8) can be expressed as [5,7]
$\phi_{A_2}(\zeta) = H_{A_1}(\zeta) + H_{A_2}(\zeta) + H_{A_3}(\zeta) + \frac{1+i\mu}{2} J_A(\zeta) + Q_A(\zeta) \chi(\zeta)$  \hspace{1cm} (10)

where

\begin{align*}
H_{A_1}(\zeta) &= P(1-i\mu) \frac{\chi(\zeta)^1}{2\pi i} \int_{\Gamma} \frac{d\sigma}{\chi(\sigma)(\sigma - \zeta)} \\
H_{A_2}(\zeta) &= \frac{G_i(1-i\mu)}{R(\kappa + 1)} \frac{\chi(\zeta)}{2\pi i} \int_{\Gamma} \frac{\{\omega(\sigma)\}^2}{\chi(\sigma)(\sigma - \zeta)} d\sigma \\
H_{A_3}(\zeta) &= \frac{1-i\mu}{2} \frac{1}{2\pi i} \left[ (\bar{q} - \kappa q) F_1 + (\kappa \bar{q} - q) F_2 + q G_1 + \bar{q} G_2 + 2\pi G_3 \right] \\
Q_A(\zeta) \chi(\zeta) &= -\sum_{k=1}^{N} \frac{\chi(\zeta)}{\chi(\zeta_k)} \frac{A_k B_k}{(\zeta \neq \zeta_k)} \\
J_A(\zeta) &= -\sum_{k=1}^{N} \left[ \frac{1}{\chi(\zeta_k)} \frac{A_k B_k}{(\zeta \neq \zeta_k)} \right] + \frac{1-i\mu}{2 i} \sum_{k=1}^{N} \left[ \frac{1}{\chi(\zeta_k)} \frac{A_k B_k}{(\zeta_k \neq \zeta)} \right] + \text{const} \hspace{1cm} (11e)
\end{align*}

and $\chi(\zeta) = (\zeta - \alpha)^{m} (\zeta - \beta)^{1-m}$, $m = 0.5 - i \ln \gamma / 2\pi$, $q = -(q_x + iq_y) / (1 + \kappa)$.

$H_{A_1}(\zeta)$ is related to the load on the punch. Though it is in integral form, its first derivative can be expressed in the form without integration \[5\]; $H_{A_2}(\zeta)$ is related to the vertical displacement (see (2c)) on the contact region induced by the radius of curvature of the punch. Owing to the use of the rational mapping function, the integration of $H_{A_2}(\zeta)$ can be carried out \[6\]; $A_k$ and $A_k$ are determined by solving 2N linear simultaneous equations for real and imaginary parts of $A_k = \phi_{A_2}(\zeta_k)$ ($k = 1, 2, ..., N$), and $H_{A_3}(\zeta)$ is related to the concentrated forces in the half plane with

\begin{align*}
F_1 &= \log(\sigma - 1/\bar{\zeta}_0) - \log(\sigma - 1/\bar{\zeta}_m) + \chi(\zeta) \int_{\bar{\zeta}_0}^{\bar{\zeta}_m} \frac{d\sigma}{\chi(\sigma)(\sigma - \zeta)} \\
F_2 &= \log(\sigma - \bar{\zeta}_0) - \log(\sigma - \bar{\zeta}_m) + \chi(\zeta) \int_{\bar{\zeta}_0}^{\bar{\zeta}_m} \frac{d\sigma}{\chi(\sigma)(\sigma - \zeta)} \\
G_1 &= \frac{\omega(\bar{\zeta}_0) - \omega(1/\bar{\zeta}_0)}{\omega(\bar{\zeta}_0)} \left[ 1 - \frac{\chi(\bar{\zeta})}{\chi(\bar{\zeta}_0)} \right] \frac{1}{\bar{\zeta} - \bar{\zeta}_0} - \frac{\omega(\bar{\zeta}_m) - \omega(1/\bar{\zeta}_m)}{\omega(\bar{\zeta}_m)} \left[ 1 - \frac{\chi(\bar{\zeta}_m)}{\chi(\bar{\zeta}_0)} \right] \frac{1}{\bar{\zeta}_m - \bar{\zeta}_0} \\
&- \sum_{k=1}^{N} \left( \frac{1}{\bar{\zeta}_k - \bar{\zeta}_0} - \frac{1}{\bar{\zeta}_k - \bar{\zeta}_m} \right) \left[ 1 - \frac{\chi(\bar{\zeta}_k)}{\chi(\bar{\zeta}_0)} \right] \frac{B_k \bar{\zeta}_k^2}{\bar{\zeta}_k - \bar{\zeta}_0} \\
G_2 &= \frac{\omega(\bar{\zeta}_0) - \omega(1/\bar{\zeta}_0)}{\omega(\bar{\zeta}_0)} \left[ 1 - \frac{\chi(\bar{\zeta})}{\chi(1/\bar{\zeta}_0)} \right] (1/\bar{\zeta}_0)^2 \frac{1}{\bar{\zeta} - 1/\bar{\zeta}_0}
4.2 The solution of part B

For part B, i.e., the flat-ended punch inclined by the moment on the contact region without load, according to the conditions described by (18), the solution of part B can be obtained as [8]

\[ \phi_B(\zeta) = H_B(\zeta) + \frac{1 + i\mu}{2} J_B(\zeta) + Q_B(\zeta) \chi(\zeta) \]  

(12)

where

\[ H_B(\zeta) = \frac{2Gi(1-i\mu)}{\kappa + 1} \int \frac{-\varepsilon \omega(\sigma)}{\chi(\sigma)(\sigma - \zeta)} d\sigma \]

= \frac{2Gei(1 - i\mu)}{\kappa + 1} \left[ \frac{\chi(\zeta)E_0}{\chi(1)(1 - \zeta)} + \sum_{k=1}^{N} \frac{E_k \chi(\zeta_k)\chi(\zeta_k - \zeta) - \omega(\zeta)}{\chi(\zeta_k)(\zeta_k - \zeta)} \right]  

(13a)

\[ J_B(\zeta) = -\sum_{k=1}^{N} \left[ 1 - \frac{\chi(\zeta_k)C_kB_k}{\chi(\zeta_k)\zeta_k - \zeta} + \frac{1 - i\mu}{1 + i\mu} \sum_{k=1}^{N} \left[ 1 - \frac{\chi(\zeta_k)C_k}{\chi(\zeta_k)\zeta_k - \zeta} \right] \frac{C_kB_k \zeta_k^2}{\zeta_k - \zeta} \right] \]

(13b)

\[ Q_B(\zeta) \chi(\zeta) = -\sum_{k=1}^{N} \frac{\chi(\zeta)C_kB_k}{\chi(\zeta_k)(\zeta_k - \zeta)} \]

(13c)

\[ C_k \] and \[ \bar{C}_k \] are determined by solving 2N linear simultaneous equations for real and imaginary parts of \( C_k = \phi_B^{'}(\zeta_k) \) \( (k = 1, 2, ..., N) \).

The final solutions of \( \phi(\zeta) \) and \( \psi(\zeta) \) of the punch problem are expressed by

\[ \phi(\zeta) = \phi_\lambda(\zeta) + \phi_B(\zeta) \]  

(14)

and due to analytical continuation

\[ \psi(\zeta) = -\phi(1/\zeta) - \frac{\omega(1/\zeta)}{\omega(\zeta)} \phi(\zeta) \]  

(15)
5 The resultant moment and inclined angle of the punch

In order to find the inclined angle of the punch, the resultant moments $R_m^A$ and $R_m^B$ for parts A and B must be decided beforehand, which are calculated by [7]

$$R_m = -\Re \left[ \int_a^b \omega'(\sigma) \frac{1}{\sigma} \frac{d\sigma}{\sigma^2} + \int_a^b \omega'(1) \frac{d\sigma}{\sigma} \right]$$  \hspace{1cm} (16)

where $i=A, B$ represent parts A and B, respectively.

The non-dimensional resultant moment for each part is defined by

for part A,

$$M_{rA} = \frac{R_m^A}{P_a}$$ \hspace{1cm} (17)

for part B,

$$M_{rB} = \frac{(\kappa + 1)R_m^B}{Gea^2}$$ \hspace{1cm} (18)

The inclined angle of the punch is determined by

$$R_m^A + R_m^B = 0$$ \hspace{1cm} (19)

i.e.

$$\frac{Gea}{P} = -(\kappa + 1) \frac{M_{rA}}{M_{rB}}$$ \hspace{1cm} (20)

Figure 2 The inclined angle of the punch with $d=a$ and $\gamma=60^\circ$
Fig. 2 shows the inclined angle of the punch when the concentrated forces are supposed to be acted in the body of the half plane, and the oblique angle of the crack is typically taken as $60^\circ$. The results of $M_{rb}$ is $-1.52594$ with $b/a = 0.5, c/a = 0.0, \mu = 0.5, \kappa = 2$ and $G\alpha^2/(PR) = 1$, respectively. After $M_{ra}$ is calculated with different positions of the concentrated forces, the corresponding inclined angle of the punch can be easily found by using (20). Since $M_{rb}$ is not changed with the position of the concentrated force, it is evident that the distributing shape of $M_{ra}$ is the same as that of the inclined angle of the punch.

6 The stress intensity factors

The stress intensity factors of the crack are calculated by [4]

$$K_{ij} - iK_{ii} = 2\sqrt{\pi\epsilon} \frac{\delta a}{\sqrt{\omega^2(\alpha)}}$$

(21)

where $\alpha = (1 - 2s + \iota i) / (1 - 2s - \iota i)$ is $\zeta$ on the unit circle corresponding to the tip of the crack (point C on the unit circle), and $\delta = -\gamma \pi / 180$ represents the angle between the x-axis and the crack. $s = \gamma / 180$, and $j = A, B$ denote parts A and B, respectively.

The non-dimensional stress intensity factors for parts A and B are defined as for part A,

$$F_{IA} + iF_{IIA} = \frac{\sqrt{a}(K_{IA} + iK_{IIA})}{P\sqrt{\pi}}$$

(22)

for part B

$$F_{IB} + iF_{IIB} = \frac{(\kappa + 1)(K_{IB} + iK_{IIB})}{Ge\sqrt{\pi a}}$$

(23)

After the stress intensity factors of parts A and B are calculated, with the known inclined angle of the punch, the non-dimensional stress intensity factors of the whole problem can be obtained by the superposition of the results in the two parts as follows:

$$F_I + iF_{II} = (F_{IA} + iF_{IIA}) + \frac{Gea}{P(\kappa + 1)}(F_{IB} + iF_{IIB})$$

(24)

Fig. 3 shows $F_I$ and $F_{II}$ with the concentrated forces acted in the body of the half plane. $F_{IB}$ and $F_{IIB}$ are $-0.03192$ and $-0.21727$ for $\gamma = 60^\circ$ with $b/a = 0.5, c/a = 0.0, \mu = 0.5, \kappa = 2$ and $G\alpha^2/(PR) = 1$, respectively.
Figure 3 The non-dimensional stress intensity factors with $d=a$

7 Conclusions

The fundamental solution of circular rigid punch on a cracked half plane subjected to concentrated forces was derived. The first derivatives of $(11a)$ and $(11f, g)$ can be expressed in the form without integration, therefore the expression of stress components does not include any integral terms so that the numerical integration is not needed for the calculation of $A_z$ (see $(11d)$), $C_z$ (see $(13c)$), stress components, stress intensity factors as well as resultant moment on the contact region. Since the circular punch is acted by a vertical load at its center, the punch is usually inclined owing to the existence of the surface crack, the frictional force and the concentrated forces. The inclined angle of the punch is obtained from eqn.$(19)$. If $V_z$ (see $(2c)$) is changed, the other shape of punch can be solved. The fundamental solution in the present paper can be progressively used to form special type of boundary element. It has many advantages since the boundary conditions are completely satisfied.
References


2. Hasebe, N., Qian, J. & Chen, Y.Z. Fundamental solutions for half plane with an oblique edge crack (submit to the present Conference).


