



A boundary element solution of Oseen flow past a solid body

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Abstract

In this paper, a boundary element formulation is proposed to solve for flow past solid bodies, incorporating the effects of inertia. This will be accomplished by the use of the Oseen representation formulae. It will also be proved that the Oseen formulation can be represented in the form of a Stokes formulation with the addition of some correcting terms, thus making the problem simpler to handle. The results at this stage are shown.

1 Introduction

The low Reynolds number flow past isolated particles of various shapes such as rigid spheres, spheroids, and cylinders have been studied quite extensively by a number of researchers with a wide variety of analytical and numerical methods being used to cover a broad range of both geometric and flow parameters. A compilation of these results has been brought together both in a monograph by Clift et al. (1978) [2] and also by Happel and Brenner [5]. Almost all of the previous numerical tests have been based upon the solution of the Stokes system of equations, which neglect completely the non-linear inertial effects.

The aim of this work is to study an Oseen flow which incorporates these inertial effects upon the body in a linear manner. This is achieved by means of a first kind Fredholm integral equation solution.

It is known that Oseen's equation gives a first order approximation to the solution of the Navier-Stokes equation, an approximation which is valid for a Reynolds number much less than unity over the whole flow field (see Batchelor [3]). It is also known that, when a single body with fore and aft symmetry is moving in a fluid at rest, the Oseen approximation gives a drag coefficient that is equal to the one given by a second order approximation of the complete equations of motion. This is a valid observation and is explained by the fact that the term of order Reynolds in the difference between the solution of Oseen equations and the second approximation to the solution of the complete equations makes no contribution whatsoever to the drag for bodies with fore and aft symmetry.

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1.1 Mathematical Formulations

We will consider a body of arbitrary shape moving in a quiescent, unbounded and incompressible fluid. The equations that describe the fluid motion exterior to the body are the steady Navier–Stokes equations,

$$-\frac{\partial p}{\partial x_i} + \mu \frac{\partial^2 u_i}{\partial x_j \partial x_j} - \rho u_j \frac{\partial u_i}{\partial x_j} = 0 \quad (1)$$

$$\frac{\partial u_i}{\partial x_i} = 0. \quad (2)$$

Historically, there have been several approximations to this equation proposed as a means to obtain information about the nature of the flow. These approximations are based on the magnitude of the Reynolds number, $R_e = \rho U L / \mu$. In particular, $R_e \ll 1$ means that the inertia force is much smaller than the viscous force, so that pressure and viscous forces are dominant in the flow field.

The Stokes flow problem is characterized by very small velocities (consequently very low R_e), such that the nonlinear convective term can be neglected. The flow can then be approximated by the motion of a fluid with zero density (or infinite viscosity), generating the linear *creeping flow* equations,

$$\mu \frac{\partial^2}{\partial x_j \partial x_j} u_i = \frac{\partial}{\partial x_i} p \quad (3)$$

$$\frac{\partial}{\partial x_i} u_i = 0. \quad (4)$$

For an external flow with onset velocity U_j past a fixed body, or the flow due to a body of arbitrary shape moving with speed U_j , Oseen (1927) [4] suggested a linearization of the Navier–Stokes equations in the form,

$$\rho(U_j \cdot \frac{\partial}{\partial x_j}) u_i = -\frac{\partial}{\partial x_i} p + \mu \frac{\partial^2}{\partial x_j \partial x_j} u_i. \quad (5)$$

The solution u_i can be viewed as a perturbation of the flow due to the presence of the obstacle. Near the body, both $(U_j \cdot \frac{\partial}{\partial x_j}) u_i$ and $(u_j \cdot \frac{\partial}{\partial x_j}) u_i$ are of the same order and both, provided that $R_e \ll 1$, will be small when compared with the viscous force. However, for a sufficiently large r , $(u_j \cdot \frac{\partial}{\partial x_j}) u_i$ remains small while $(U_j \cdot \frac{\partial}{\partial x_j}) u_i$ is responsible for creating inertia forces comparable with viscous forces.

To deal with the different behaviour of the Navier–Stokes equations close and far from the boundaries, it is possible to use the mathematical technique of matched asymptotic expansions. In this way, at low Reynolds number the Stokes approximation is really a first-order problem close to the boundaries (inner problem) arising out of a perturbation-type solution to the Navier–Stokes equations. It is known that this kind of perturbation is not able to satisfy all the required boundary conditions, and the inability of this type of solution to match the required boundary conditions renders the perturbation singular. In

two dimensions the difficulty appears immediately, whereas in three dimensions the difficulty is postponed to the second term in the expansion; this is usually referred to, in the literature, as the Whitehead's paradox (see, *e.g.* Happel and Brenner, 1973) [5]. The origin of this paradox was pointed out by Oseen in 1910 who suggested a scheme for its solution (for more details see Oseen, 1927 [4]). Mathematical difficulties encountered with the solution of the inner problem are overcome by matching these solutions with the corresponding approximations far from the boundaries (outer problem), whose first-order approximation at low Reynolds number flow is just Oseen's approximation. In this way, a uniformly valid expansion is produced which is valid for small Reynolds numbers (for more details about this technique, see the original papers of Proudman and Pearson, 1957 [6], or Brenner and Cox, 1963 [7]).

The solution of the Oseen's problem yields a uniformly valid first approximation to the solution of the complete non-linear problem for two and three dimensional problems. Rigorous results on the existence of solution for the Navier-Stokes problem (Finn (1965) [8]) considering the solution as a regular perturbation whose terms are the solution of certain Oseen's problem, essentially substantiate the above claim.

1.2 Oseen Integral Equation Solution

The reduction of the solution of Oseen's system of equations to the solution of a system of integral equations was originally given by Oseen himself (1927), [4] and a complete account of the properties of the surface potentials originated from Oseen's integral representation formulae was given by Miranda and Power (1983) [9].

By the same classical methods we obtain the following integral representation formulae for the Oseen velocity field in the form of Single-layer, Double-layer and Mixed layer potentials,

$$\begin{aligned} u_k(x) = & - \int_S \sigma'_{ij}(\vec{v}^k(x, y), g^k(x, y))_y u_i(y) n_j(y) dS_y \\ & + \int_S v_i^k(x, y) \sigma_{ij}(\vec{u}(y), p(y)) n_j(y) dS_y \\ & - \rho U \int_S v_i^k(x, y) u_i(y) n_j(y) \delta_{1j} dS_y \end{aligned} \quad (6)$$

where

$$v_i^k(x, y) = -\frac{1}{4\pi\mu r} e^{K(x_1 - y_1 + r)} \delta_{ik} + \frac{\partial \phi^k(x, y)}{\partial x_i} \quad (7)$$

with

$$\phi^k(x, y) = \frac{1}{4\pi\rho U} \left(1 - e^{K(x_1 - y_1 + r)} \right) \frac{\partial}{\partial x_k} \ln(x_1 - y_1 + r) \quad (8)$$

being the fundamental singular solution of the Oseen system of equations, known as Oseenlet, at the point y and orientated in the k -th direction and where

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$$K = -\rho U/2\mu.$$

The pressure field due to an Oseenlet is given by,

$$g^k(x, y) = -\frac{1}{4\pi} \frac{(x_k - y_k)}{r^3} = \frac{1}{4\pi} \frac{\partial}{\partial x_k} \left(\frac{1}{r} \right). \quad (9)$$

Equation (6) can be rewritten in terms of two potentials only by incorporating the Mixed-layer potential into the Double-layer potential.

$$\begin{aligned} u_k(x) = & - \int_S \left[\sigma'_{ij}(\bar{v}^k(x, y), g^k(x, y))_y \right. \\ & \left. + \rho U v_i^k(x, y) \delta_{1j} \right] u_i(y) n_j(y) dS_y \\ & + \int_S v_i^k(x, y) \sigma_{ij}(\bar{u}(y), p(y)) n_j(y) dS_y. \end{aligned} \quad (10)$$

The above integral representation formulae is written for an exterior domain and is valid as long as the sought solution $\bar{u}$ converges to zero like $|x|^{-1}$ as $|x| \rightarrow \infty$ and $p(x)$ converges to zero like $|x|^{-2}$.

If we let $x \rightarrow \xi \in S$, then equation (10) can be written as,

$$\begin{aligned} \frac{1}{2} U_{k(\xi)} = & - \int_S \left[\sigma'_{ij}(\bar{v}^k(x, y), g^k(x, y))_y \right. \\ & \left. + \rho U v_i^k(x, y) \delta_{1j} \right] U_i(y) n_j(y) dS_y \\ & + \int_S v_i^k(x, y) \sigma_{ij}(\bar{u}(y), p(y)) n_j(y) dS_y \end{aligned} \quad (11)$$

where use has been made of the continuity property of the Single-layer and Mixed-layer potentials and the discontinuity properties of the Double-layer potential. Equation (11) together with the prescribed boundary velocity, $\bar{U}$, gives a first kind Fredholm integral equation for the unknown surface tractions, $f_i = \sigma_{ij}(\bar{u}, \rho) n_j$.

In the formulation of the Oseen integral representation formulae it can be proved that the singular behaviour of the integrals appearing in the above equation are at most weakly singular. However, apparent singularities of $O(r^{-2})$ and $O(r^{-3})$ will appear when $r \rightarrow 0$ if expressions (7) and (8) are used. This problem will be addressed in the next section.

1.3 Numerical Analysis

In order to simplify the Oseen's kernel, which is quite complex, we will generalize an idea of Brenner (1961) [1], who showed how the Oseen resistance of a single particle of arbitrary shape may be obtained from its Stokes resistance (for more details see Power et. al. [10]). Following this procedure it is found that, after

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some manipulations, the Oseenlet can be written in terms of the Stokeslet, as follows,

$$v_i^k(x, y) = u_i^k(x, y) + C_{ik}(x, y), \quad (12)$$

where $u_i^k(x, y)$ is the fundamental solution of the permanent Stokes equation (Stokeslet),

$$u_i^k(x, y) = \frac{1}{8\pi\mu r} (\delta_{ik} + r_{,i} r_{,k}), \quad (13)$$

and C_{ik} is an Oseen correcting function given by,

$$\begin{aligned} C_{ik}(x, y) = & \frac{K}{8\pi\mu} \left\{ r_{,i} r_{,k} \left(F'_{(z)} - \frac{F(z)}{Kr} \right) + F'_{(z)} (\delta_{1i} r_{,k} + \delta_{1k} r_{,i} + \delta_{1i} \delta_{1k}) \right. \\ & \left. + \delta_{ik} \left[(1 - 2z) \frac{F(z)}{Kr} + \frac{2z}{Kr} \right] \right\}, \end{aligned} \quad (14)$$

where,

$$z = K(x_1 - y_1 + r),$$

$$F(z) = \frac{(1 - e^z) + 1}{z}, \quad (15)$$

$$F'_{(z)} = - \frac{(ze^z + 1 - e^z)}{z^2}, \quad (16)$$

$$r_{,l} = \frac{\partial r}{\partial x_l}.$$

The Double-layer potential kernel can also be rewritten,

$$\sigma'_{ij}(\mathbf{r}^*, g^k) n_j = P_{ik}(x, y) = K_{ik}(x, y) + e_{ik}, \quad (17)$$

where $K_{ik}(x, y)$ is the surface traction of the Stokeslet, known as the Stresslet,

$$K_{ik}(x, y) = \frac{3(r_{,i} r_{,k})}{4\pi r^2} \frac{\partial r}{\partial n_y}, \quad (18)$$

and e_{ik} , an oseen correcting function given by,

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$$\begin{aligned}
 e_{ik} = & \frac{K}{8\pi} \left\{ \left[\frac{2}{r} \frac{\partial r}{\partial n} (3r_{,i}r_{,k} - \delta_{ik}) - \frac{1}{r} (r_{,i}n_k + r_{,k}n_i) - \frac{2(1-z)}{r} (r_{,i}n_k + r_{,k}n_i) \right. \right. \\
 & - 2 \left(\frac{\partial z}{\partial x_i} n_k + \frac{\partial z}{\partial x_k} n_i \right) \left. \right] \frac{F}{Kr} + \left[\frac{1}{r} \left(2\delta_{ik} + r_{,k}n_i + r_{,i}n_k - 4 \left(r_{,i}r_{,k} \frac{\partial r}{\partial n} \right) \right. \right. \\
 & + \delta_{1k} \left(\frac{n_i}{r} - \frac{1}{r} \frac{\partial r}{\partial n} r_{,i} \right) + \delta_{1i} \left(\frac{n_k}{r} - \frac{1}{r} \frac{\partial r}{\partial n} r_{,k} \right) + 2n_1 \left(\frac{\delta_{ik}}{r} - \frac{1}{r} r_{,i}r_{,k} \right) \\
 & + \left(\frac{1-2z}{Kr} \right) \left(\frac{\partial z}{\partial x_i} n_k + \frac{\partial z}{\partial x_k} n_i \right) - r_{,k} \frac{\partial r}{\partial n} \frac{\partial z}{\partial x_i} \frac{1}{Kr} - r_{,i} \frac{\partial r}{\partial n} \frac{\partial z}{\partial x_k} \frac{1}{Kr} \left. \right] F' \\
 & + \left[\left(n_1 + \frac{\partial r}{\partial n} \right) \left((\delta_{1k} + r_{,k}) \frac{\partial z}{\partial x_i} + (\delta_{1i} + r_{,i}) \frac{\partial z}{\partial x_k} \right) \right] F'' \\
 & \left. - \frac{2z}{Kr^2} (r_{,i}n_k + r_{,k}n_i) + \frac{2}{Kr} \left(\frac{\partial z}{\partial x_i} n_k + \frac{\partial z}{\partial x_k} n_i \right) \right\} \quad (19)
 \end{aligned}$$

where,

$$\begin{aligned}
 F'' &= \left[\frac{2(z e^z + 1 - e^z) - z^2 e^z}{z^3} \right] \quad (20) \\
 \frac{\partial z}{\partial x_l} &= K (\delta_{1l} + r_{,l}).
 \end{aligned}$$

Since the functions F , F' and F'' can be expanded in a power series for small z as,

$$\begin{aligned}
 F_{(z)} &= - \sum_{l=1}^{\infty} \frac{z^l}{(l+1)!}, \\
 F'_{(z)} &= - \sum_{l=1}^{\infty} l \frac{z^{(l-1)}}{(l+1)!}, \\
 F'' &= - \sum_{l=2}^{\infty} l(l-1) \frac{z^{(l-2)}}{(l+1)!}, \quad (21)
 \end{aligned}$$

it follows that the integrals in equation (11) are at most weak singular as $r \rightarrow 0$. Therefore, the above decomposition enables us to avoid the apparent singularity of the original formulation.

2 Results

To test the numerical method developed in the present work, the problem of a translating sphere of unit radius was calculated, whose analytical solution is,

$$F_D = 6\pi a\mu U \left(1 + \frac{3\rho aU}{8\mu} \right) \quad (22)$$

where F_D represents the total drag force in the system.

A Comparison of the total force found by the above numerical solution with the analytical solution is presented in figure (1). It can clearly be seen that the results are in good agreement.

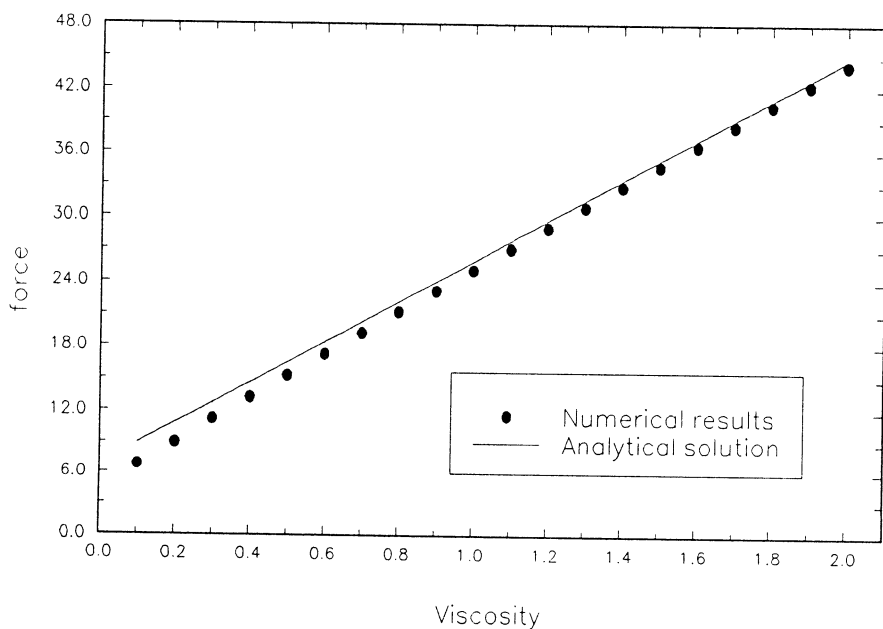


Figure 1: Drag force on a solid sphere of unit radius.

The example defined was discretised into 56 elements and was calculated using 12 Gaussian points. It is possible to increase the accuracy of this solution by increasing the number of Gaussian integration points used in the calculation and the number of elements in the mesh.



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References

- [1] Brenner, H. - "The Oseen resistance of a particle of arbitrary shape", *J. Fluid M.*, vol. 11, 1961.
- [2] Clift, R., Grace, J.R. and Weber, M.E. (1978) *Bubbles, Drops, and Particles*, Academic Press, New York.
- [3] Batchelor G.K. - *An Introduction to Fluid Dynamics*, Cambridge Univ. Press, Cambridge, 1967.
- [4] Oseen, C.W. (1927) *Neuere Methoden und Ergebnisse in der Hydrodynamik*, Akademische Verlag, Leipzig.
- [5] Happel, J. and Brenner, H. (1973) *Low Reynolds Number Hydrodynamics*, Martinus Noordhoff, The Netherlands.
- [6] Proudman, I. and Pearson, J.R.A. (1957) Expansions at small Reynolds numbers for the flow past a sphere and a circular cylinder, *Journal of Fluid Mechanics*, Vol. 2, 237-262.
- [7] Brenner, H. and Cox, R.G. (1963) The resistance to a particle of arbitrary shape in translational motion at small Reynolds numbers, *Journal of Fluid Mechanics*, Vol. 17, 561-595.
- [8] Finn, Finn, R. (1965) "On the exterior stationary problem for the Navier-Stokes equations, and associated perturbation problems" *Arch. Rational Mech. Anal.* 19.
- [9] Miranda, G. and Power, H., (1983) Integral equation solution of Oseen's flow with a free surface, *Lecture Notes in Mathematics, Numerical Methods*, No. 1005, Springer-Verlag, Berlin, 170-203.
- [10] Power, H., Miranda, G. and Villamizar, V, (1985) Integral equation solution for the flow due to the motion of two bodies at small Reynolds number. *Mat. Aplic. Comp.* 4, 49-66.