Transactions on Modelling and Simulation vol 9, © 1995 WIT Press, www.witpress.com, ISSN 1743-355X

An introduction to the boundary element method in electromagnetism: physical basis and applications

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Abstract

This introductory paper presents the physical interpretation of the boundary element method in electromagnetism. Electrostatics is taken as a basic example and a capacitive circuit analogy is introduced to demonstrate some principles.

1 Introduction

The boundary element method is used in several fields of engineering. It has now proved to be an important tool for numerical computations. But though many scientists have some knowledge on finite element methods they usually know very little about the boundary element method. This is maybe due to the mathematical tools required to enter the boundary element world: Green function, Green identity, fundamental solution, singular integrals ... Nevertheless, the boundary element method is quite natural in electromagnetism and some of the classical tools of electricians are precisely what is required for BEM. Moreover, using electric circuit analogy allows to present a simplified version of BEM without singular kernels.

2 Electrostatics

It has been found experimentally that the force $\mathbf{F}_{12}$ between two stationary electric point charges q_1 and q_2 is given by the Coulomb's law [1]:

$$\mathbf{F}_{12} = \frac{1}{4\pi\varepsilon_o} \frac{q_1 q_2}{r^3} \mathbf{r} \tag{1}$$

where **r** is the vector from 1 to 2 and r its norm. This gives the force acting on q_2, whereas the force acting on q_1 is the opposite. ε_o is the permittivity of free space. This can be genelarized to linear dielectric media by taking a constant permittivity ε characteristic of the media instead of ε_o. In this classical and simple formula, the singularity problem is already present when the force of a point charge on itself is considered.

Now let introduce field concept. The field $\mathbf{E}_1$ produced by the charge q_1 is such that the force on q_2 is given by $q_2\,\mathbf{E}_1$. The expression for this field is :

$$\mathbf{E}_1 = \frac{\mathbf{F}_{12}}{q_2} = \frac{1}{4\pi\varepsilon} \frac{q_1}{r^3} \mathbf{r} \tag{2}$$

In fact, the total electric field is the sum of the field due to q_1 plus the field due to q_2. Therefore the actual electric field is defined by introducing a test charge in the system an taking the limit of the ratio of the force on the charge when this charge decreases to zero. The general rule for obtaining the field from a set of charges is a simple addition of the contributions. For a distribution of charges characterized by a density ρ, this principle of superposition leads to the integral formula:

$$\mathbf{E} = \int_{R^3} \frac{1}{4\pi\varepsilon} \frac{\rho}{r^2} \mathbf{r} \, dV \tag{3}$$

Conversely, the passage from (3) to (2) is made by choosing a charge distribution $\rho = q_1\,\delta(\mathbf{r})$ where $\delta(\mathbf{r})$ is the Dirac distribution (in the Schwartz distribution sense [2]). Another vector field is usually introduced which is the electric flux density $\mathbf{D} = \varepsilon * \mathbf{E}$ and which is related to the charge density by div $\mathbf{D} = \rho$. The relation between **E** and **D** involves not only the characteristic ε of the media but also an operator *, the Hodge star operator, usually hidden in vector analysis, that transforms a circulation field into a flux density [3]. Here, * is its own inverse i.e. ** = 1.

If one is interested in the work done by a charge displacement, it is given by the line integral of the force along the path. If the charge remains constant along the path, it can be expressed as the product of the charge multiplied by the line integral of the field $q\int_{path} \mathbf{E}\, dl$. It appears that for given starting and arriving points, this quantity does not depend on the path. Once a starting point is chosen, it defines a function V of the position. The freedom of choice on the

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starting point is related to the fact that a reference for the total energy can be choosen. The quantity V is the electric potential and the electric field is retrieved by taking its gradient **E** = - grad V. By now, we have defined the two fundamental quantities of electrostatics: the charge density ρ and the electrostatic potential V that gives the electrostatic potential energy density $\frac{1}{2}\rho V$ (see reference [1] for an explanation of the $\frac{1}{2}$ factor). Using (3) and the fact that **E** = -grad V, we find:

$$V = \int_{R^3} \frac{1}{4\pi\varepsilon}\frac{\rho}{r}\, dV = \frac{1}{\varepsilon}\int_{R^3} \rho\ G\ dV = K\ \rho \qquad (4)$$

where the singular kernel G = $1/4\pi r$ is the Green function of the Laplace operator Δ involved in the equation $\Delta V = -\rho/\varepsilon$. Therefore the integral operator K acting on ρ in (4) is the inverse operator of the Laplace differential operator. G is also called the fundamental solution because it corresponds to the potential generated by $\rho = \varepsilon\ \delta(\mathbf{r})$. This is elementary electrostatics and it is summarized on figure 1 where the operators (the differential ones and their integral inverses) connecting the electrostatic quantities are given.

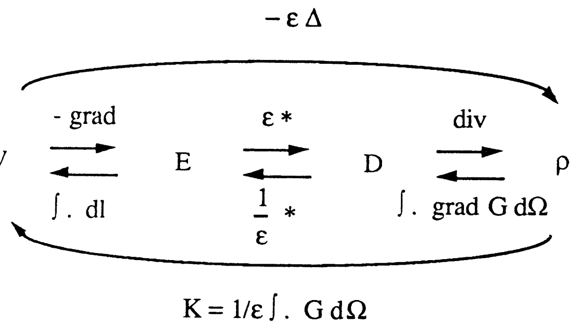

Figure 1: operators from V to ρ and their inverses.

All the elements of the boundary element method [4] are present in elementary electrostatics. The only new element required is the Green identity:

$$\int_{\Omega} (u\,\Delta v - v\,\Delta u)d\Omega = \oint_{\partial\Omega} (u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n})d\partial\Omega \qquad (5)$$

where $\partial\Omega$ is the boundary of the domain Ω , $\partial./\partial n$ is for the normal derivative, u and v are arbitrary functions. Introducing V and G in (5) gives:

$$hV = \int_{\Omega} \frac{\rho}{\varepsilon} G \, d\Omega + \oint_{\partial\Omega} \left(G\frac{\partial V}{\partial n} - V\frac{\partial G}{\partial n}\right) d\partial\Omega \tag{6}$$

where h is a coefficient depending on the observation point (h=1 inside Ω, 0 outside and .5 on a smooth part of the boundary). Before the emergence of BEM as a powerful computation tool, expression (6) was known as the 'three potential theorem'. The three terms of the right-hand member of (6) can be interpreted as the potential of three kinds of sources, respectively volume sources inside V, a single layer of charges on the boundary of Ω (surface density of charges), and a double layer of charges on the boundary of Ω (surface density of dipoles). The first term is the usual contribution of charges inside Ω while the two other terms take into account the influence of the outside of the domain. They provide an equivalent situation where the outside of Ω is empty but where the single layer provides the same potential on the boundary and the double layer provides the same electric flux across the boundary with respect to the actual situation.

The boundary element method based on formula (6) is called the 'direct boundary element method'. In some problems where the potential is continuous e.g. involving dielectrics and perfect conductors, it is possible to find an equivalent situation by placing single layers of charges on the discontinuity surfaces of the flux density (i.e. the interface between media). This is called the 'indirect boundary element method'. This can be modified further by placing charge densities elsewhere than on the interfaces and even by using point charges. This is the 'charge simulation method', a particular case of which is the 'image method' [1]. These methods were used by electricians long before the appearance of computers and are now traditional methods in electrostatics. All these methods, including BEM, rely on equivalent charges and their remote action (thanks to Green function). It demonstrates that the BEM is natural and sounds familiar to the electricians.

3 Capacitive circuit analogy

Capacitive circuits are a discrete case of electrostatics where some features of the boundary element method can be easily demonstrated. The advantage here is that all the operators are matrices instead of differential or integral operators and that we get rid of singularities.

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The starting point is the study of a simple perfect capacitor (figure 2) where the electric field is supposed to be constant in the parallelipipedic space (dx, dy, dz) between the plates and equal to zero outside. The capacitor connects a point A of a circuit to a point B and constitutes the branch α. This is expressed with the help of a boundary operator ∂ as $\partial\alpha$=B-A; the minus sign is associated to the origin [6].

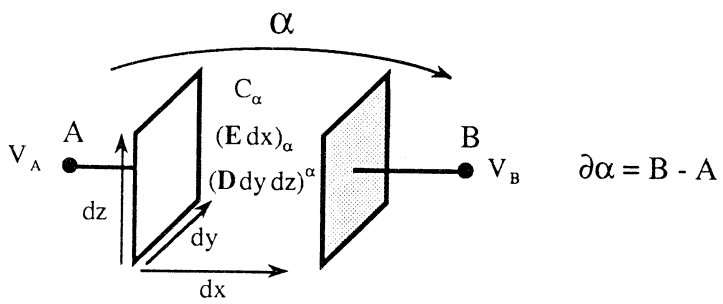

Figure 2: perfect capacitor

Field	**Circuit**
V	V_A
$E = -\operatorname{grad} V$	$(E\ dx)_\alpha = V_B - V_A$
$D = \varepsilon * E$	$(D\ dy\,dz)^\alpha = Q^\alpha$ $= \varepsilon \frac{dy\ dz}{dx}(V_B - V_A) = C_\alpha(V_B - V_A)$
$\operatorname{div} D = \rho$	$Q^A = \sum_{\alpha:A\in\partial\alpha} \pm Q^\alpha$

Table 1 : local equations for the capacitive network

Table 1 gives the local equations for the capacitive circuit deduced from the corresponding field relations. The first line indicates that the potential V_A at each node A is considered. The line integral of the electric field across the capacitance is just the product of the electric field by the distance dx between the plates and is equal to the difference of potential between the plates. The constitutive equation corresponds to the capacitance definition. The electric charges of opposite signs stored on the plates are equal to the electric flux between the plates. This flux is the product of the electric flux density by the section dy.dz of the capacitor. It appears that the charge is proportional to the difference of potential. The constant of proportionality is the capacitance. It is the product of the permittivity by a factor dy.dz/dx depending on the geometry. This second factor corresponds to the Hodge operator: it translates the line integral along dx into a flux through dy.dz. Here, in the language of differential forms, the analogy with the Hodge operator appears clearly [3,6]. Note that only this line of the table depends on metric aspects, all other lines depend only on the topology of the circuit. The last line defines a charge associated to nodes which is the sum of all the charges on the plates connected to the node with their signs. By analogy with electrostatics the fundamental quantities are now defined as nodal potentials and charges.

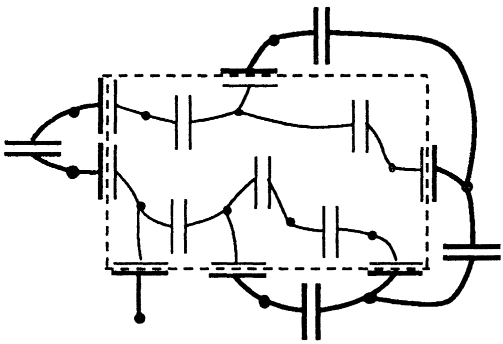

Figure 3 : capacitive network

If a complete capacitive network (figure 3) is considered with N nodes connected by B branches, all the relations can be gathered in matrix form. The following vectors and matrices are considered:

vector/matrix	definition	dimension
V	vector of nodal values of the potential	N
E	vector of line integral of the field along branches	B
D	vector of electric fluxes through branches	B
Q	vector of nodal values of charges	N
A	incidence matrix such that E = A V	B x N
C	capacitance matrix such that D = C E	B x B
A^T	nodal matrix such that $Q = A^T D$	B x N

Each line of A contains one term equal to 1 and one term equal to -1 for the boundary nodes of the corresponding branch. All the other terms are equal to zero. Each line of A^T contains 1 or -1 for the branches connected to the corresponding node. They are topological matrices of the network and are obviously the transpose of each other. C is the diagonal matrix of the capacitances. The relation between V and Q is:

$$Q = A^T C A V = \Delta V \tag{7}$$

where Δ is a symmetric matrix corresponding to the Laplace operator. The inverse operator is just the matrix $K = \Delta^{-1}$ (Strictly speaking, Δ is singular because several potential configurations correspond to the same charges and we have to fix the potential at one node per connected part of the circuit and eliminate the corresponding part of the matrices [6]. This is beyond the scope of this elementary paper and we consider that this problem is solved). An element K_{AB} of matrix K can be interpreted as the potential generated at node B if the charge on node A is equal to 1 and the charges on all the other nodes are equal to zero. The analogy with Green function is obvious.

Two different potential vectors u and v are considered and a bilinear symmetric form (u,v) is constructed with Δ:

$$(u,v) = (v,u) = u^T \Delta v = v^T \Delta u \tag{8}$$

This form can be interpreted as the scalar product of the potential vector u with the charge vector $q = \Delta v$ corresponding to the potential vector v. To perform the computation, the vector v is transformed into a corresponding charge vector by the matrix Δ and the scalar product of this new vector is taken with vector u i.e. a sum of the product of the corresponding components is computed. Therefore this sum can be decomposed in the contribution of the interior nodes (thin part of figure 3), written $(u,v)_\Omega$, and the contribution of the boundary nodes (bold part of figure 3), written $(u,v)_{\partial\Omega}$. Now the trivial equality $(u,v)-(v,u) = 0$ can be written:

$$(u,v)_\Omega - (v,u)_\Omega = (v,u)_{\partial\Omega} - (u,v)_{\partial\Omega} \tag{9}$$

In order to modify the right-hand member, we have to compute the charge q^A on a boundary point A:

$$
\begin{aligned}
q^A = (\Delta u)^A &= \sum_{\alpha:\partial\alpha=\pm(B-A)} C_\alpha (u_A - u_B) \\
&= \sum_{\substack{\alpha:\partial\alpha=\pm(B-A)\\ B \text{ interior}}} C_\alpha (u_A - u_B) + \sum_{\substack{\alpha:\partial\alpha=\pm(B-A)\\ B \text{ on the boundary}}} C_\alpha (u_A - u_B) \\
&= \sum_{\substack{\alpha:\partial\alpha=\pm(B-A)\\ B \text{ interior}}} C_\alpha (u_A - u_B) + (\Delta^\partial u)^A
\end{aligned}
\tag{10}
$$

The contribution of boundary points has been distinguished. The boundary is a circuit on its own right and the corresponding Laplace operator matrix Δ^∂ has been introduced in (10). The right-hand member of (9) becomes :

$$
\begin{aligned}
(v,u)_{\partial\Omega} - (u,v)_{\partial\Omega} &= \sum_{A \text{ on the boundary}} u_A (\Delta v)^A - v_A (\Delta u)^A = \\
&\sum_{A \text{ on the boundary}} u_A \Big(\sum_{\substack{\alpha:\partial\alpha=\pm(B-A)\\ B \text{ interior}}} C_\alpha (v_A - v_B) + (\Delta^\partial v)^A \Big) - v_A \Big(\sum_{\substack{\alpha:\partial\alpha=\pm(B-A)\\ B \text{ interior}}} C_\alpha (u_A - u_B) + (\Delta^\partial u)^A \Big) \\
&= \sum_{A \text{ on the boundary}} u_A \Big(\sum_{\substack{\alpha:\partial\alpha=\pm(B-A)\\ B \text{ interior}}} C_\alpha (v_A - v_B) \Big) - v_A \Big(\sum_{\substack{\alpha:\partial\alpha=\pm(B-A)\\ B \text{ interior}}} C_\alpha (u_A - u_B) \Big)
\end{aligned}
\tag{11}
$$

The last line has been obtained thanks to the symmetry of Δ^∂ that leads to a relation similar to (8). Introducing this line in (9) gives an analogue to the Green identity (5) [6]:

$$
\begin{aligned}
&\sum_{A \text{ interior}} u_A (\Delta v)^A - v_A (\Delta u)^A = \\
&\sum_{A \text{ on the boundary}} u_A \Big(\sum_{\substack{\alpha:\partial\alpha=\pm(B-A)\\ B \text{ interior}}} C_\alpha (v_A - v_B) \Big) - v_A \Big(\sum_{\substack{\alpha:\partial\alpha=\pm(B-A)\\ B \text{ interior}}} C_\alpha (u_A - u_B) \Big)
\end{aligned}
\tag{12}
$$

The analogy between the left-hand members of (5) and (12) is obvious. As for the right-hand member of (12), it is a sum of contributions of points on the boundary ('boundary integral') and the differences are computed with respect to interior points only ('normal derivative').

The analogue of the boundary element method can be found by introducing the appropriate vectors in (12). An interior point C is choosen. The vector v is choosen as the column C of matrix K whose components are K_{AC}. To compute the first term, the fact that K is the inverse of Δ is used.

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$$u_C = \sum_{A \text{ interior}} K_{AC} q^A + \sum_{A \text{ on the boundary}} u_A \Big(\sum_{\substack{\alpha:\partial\alpha=\pm(B-A) \\ B \text{ interior}}} C_\alpha (K_{AC} - K_{BC}) \Big) - K_{AC} \Big(\sum_{\substack{\alpha:\partial\alpha=\pm(B-A) \\ B \text{ interior}}} C_\alpha (u_A - u_B) \Big) \tag{13}$$

The expression is similar to (6) and can be considered as the circuit analogue of the boundary element method.

4 Electromagnetism

As electromagnetism is just electrostatics together with special relativity, it is not amazing that the boundary element method naturally emerges in the whole field. For instance, the remote action of sources and the corresponding Green function are used in magnetostatics. The vector potential **A** produced by a current density **J** in a conductor is given by:

$$\mathbf{A} = \mu_o \int_{\text{conductor}} \frac{\mathbf{J}}{4\pi r} \, dV \tag{14}$$

The curl of (13) gives the well known Biot-Savart law for the magnetic flux density [6].

For electromagnetic waves, the analogues of (4) and (14) are the formulae for the Lienard-Wiechert retarded potentials [6] involving the Green function of the d'Alembertian operator.

The BEM has also been applied to other problems such as the eddy current problems or axisymmetric problems where the physical interpretation of the Green function is more or less obvious [7].

5 Conclusion

The physical basis of the boundary element method in electromagnetism appears clearly. Nevertheless the use of the boundary element method as a numerical method requires a lot of work. The first problem encountered is the choice of the discretisation method: collocation, Galerkin, ... Then several difficult numerical problems have to be solved such as the computation of the integrals with singular or quasi singular kernels and the solution of large full algebraic systems. Moreover, the method has to be modified to cope with non linear problems. An obvious way is to couple the method with the finite element method [7] and in this case several variants are possible. Another new promising method is the dual reciprocity boundary element method recently applied to magnetic problems [8].

Once all those problems have been solved, the resulting numerical model presents several interesting features. The boundary element method provides a smooth solution inside the domain and is often more accurate than the piecewise polynomial solutions of finite element methods. The open problems with boundary conditions at infinity are automatically taken into account. Such problems are very common in electromagnetism where fields propagate in air and free space. Free boundary problems or problems involving movements are easily taken into account because remeshing is much easier or even unnecessary. In electromagnetism, this is a very useful property for the modelling of electromechanical converters such as motors or relays.

Therefore, considering those advantages of the boundary element method, it is worth the work.

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