Application of Trefftz method to 3D potential problem and its sensitivity analysis
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Abstract

This paper presents a new boundary-type scheme for the sensitivity analysis of three-dimensional potential problem by Trefftz formulation, which is formulated by the collocation method. Then, the shape sensitivity analysis scheme based on the collocation-Trefftz method is explained. The heat transfer on a thick-walled cylinder is considered as a numerical example in order to confirm the validity of the formulations.

1 Introduction

Sensitivity analysis scheme is very significant technique for solving various inverse problems such as the shape optimization, the moving boundary problem and so on. Therefore, many researchers already presented the sensitivity analysis schemes based on the finite or the boundary element method. However, they have some difficulties. Since the finite element method is the domain-type solution procedure, input data generation is very difficult. Besides, since the interpolating function depends on the shape parameters controlling the boundary profile of the object region, the sensitivities, which is derived from differentiation of physical quantities, is approximated by lower-order interpolating function than the original quantities. On the other hand, in the boundary element method, the singular property of the integral equation is great difficulty. Since the physical quantities are expressed by the singular integral equation, the sensitivities is expressed by
56 Boundary Element Technology

hyper-singular integral equation and therefore, special techniques are necessary for formulating the hyper-singular equations[1-4]. These difficulties can be overcome by the Trefftz method.

The Trefftz method is one of the boundary-type solution procedures based on the regular formulation. Therefore, the input data generation is easier than the finite element method. Moreover, since the physical quantities are given in terms of the regular expressions, the differentiation of the quantities leads to the regular expressions of the sensitivities. Besides, the physical quantities are approximated by the superposition of the regular T-complete functions satisfying the governing equation. Since the T-complete function is independent of the shape variables, the order of the T-complete functions is not reduced by the differentiation of the quantities.

By the way, the Trefftz method is not yet applied to the three-dimensional potential problem. Therefore, in this paper, firstly the basic solver for the three-dimensional potential problem will be formulated by the collocation method and then, the sensitivity will be formulated. A heat transfer in a thick-walled cylinder is considered as a numerical example in order to confirm the validity of the present formulations.

2 Trefftz Method

2.1 Governing equation and T-complete functions

The governing equation and the boundary condition of the three-dimensional potential problem are given as

$$\nabla^2 u = 0 \quad \text{(in } \Omega)$$

and

$$u = \bar{u} \quad \text{(on } \Gamma_1)$$

$$q \left( \equiv \frac{\partial u}{\partial n} \right) = \bar{q} \quad \text{(on } \Gamma_2)$$

where $\Omega$, $\Gamma_1$ and $\Gamma_2$ denote the region occupied by the object, its potential-specified and flux-specified boundaries, respectively. $n$ denotes the normal vector on the boundary and $(\bar{\cdot})$ the specified value on the boundary.

The T-complete function for the three-dimensional potential problem in the bounded region is given as[5, 6]

$$u^* = \left\{ r^\nu P_\nu^\mu (\cos \theta) e^{j\mu \phi} \right\} \quad (\nu = 0, 1, 2, \cdots; \mu = 0, 1, \cdots \nu)$$

where $r$, $\theta$ and $\phi$ are the spherical coordinates whose origin is taken arbitrarily and $j$ denotes the imaginary unit. $P_\nu^\mu$ denotes the Legendre associated function.
2.2 Collocation formulation

The potential value $u$ is approximated by the superposition of the T-complete functions $u_i^*$ as follows

$$ u \simeq \tilde{u} = a_1 u_1^* + a_2 u_2^* + \cdots + a_N u_N^* \equiv a^T u^* $$

(4)

where $a = \{a_1, a_2, \ldots, a_N\}^T$ denotes the unknown parameter vector and (\tilde{}) means approximate solution. $N$ is the total number of the T-complete functions. Differentiating Eq.(4) in the normal direction, we have

$$ q \simeq \tilde{q} \left( \equiv \frac{\partial \tilde{u}}{\partial n} \right) = a_1 q_1^* + a_2 q_2^* + \cdots + a_N q_N^* \equiv a^T q^* $$

(5)

Equations (4) and (5) do not satisfy Eq.(2). Therefore, the residuals yield;

$$ R_1 \equiv \tilde{u} - u = a^T u^* - \tilde{u} \neq 0 \quad \text{on} \quad \Gamma_1 $$

$$ R_2 \equiv \tilde{q} - q = a^T q^* - \tilde{q} \neq 0 \quad \text{on} \quad \Gamma_2 $$

(6)

$a$ is determined so that the approximate solution satisfies the boundary condition. In the collocation method, the residuals are forced to vanish at the boundary collocation point $P_m$. From Eq.(6), we have

$$ R_1(P_m) = a^T u^*(P_m) - \tilde{u}(P_m) = 0 \quad (P_m \text{ on } \Gamma_1, m = 1, \ldots, M_1) $$

$$ R_2(P_m) = a^T q^*(P_m) - \tilde{q}(P_m) = 0 \quad (P_m \text{ on } \Gamma_2, m = 1, \ldots, M_2) $$

(7)

where $M_1$ and $M_2$ are the total numbers of the collocation points on the boundaries $\Gamma_1$ and $\Gamma_2$, respectively. Rearranging the above equation, we have

$$ a^T u^*(P_m) = \tilde{u}(P_m) \quad (P_m \text{ on } \Gamma_1) $$

$$ a^T q^*(P_m) = \tilde{q}(P_m) \quad (P_m \text{ on } \Gamma_2) $$

(8)

or

$$ \begin{bmatrix}
  u_{11} & u_{12}^* & \cdots & u_{1N}^* \\
  u_{21} & u_{22}^* & \cdots & u_{2N}^* \\
  \vdots & \vdots & \ddots & \vdots \\
  u_{M_11} & u_{M_12}^* & \cdots & u_{M_1N}^* \\
  q_{11}^* & q_{12}^* & \cdots & q_{1N}^* \\
  q_{21}^* & q_{22}^* & \cdots & q_{2N}^* \\
  \vdots & \vdots & \ddots & \vdots \\
  q_{M_21}^* & q_{M_22}^* & \cdots & q_{M_2N}^*
\end{bmatrix}
\begin{bmatrix}
  a_1 \\
  a_2 \\
  \vdots \\
  a_N
\end{bmatrix}
= \begin{bmatrix}
  \tilde{u}_1 \\
  \tilde{u}_2 \\
  \vdots \\
  \tilde{u}_{M_1}
\end{bmatrix} $$

(9)

$$ \begin{bmatrix}
  q_{11}^* \\
  q_{12}^* \\
  \vdots \\
  q_{M_21}^*
\end{bmatrix}
\begin{bmatrix}
  a_1 \\
  a_2 \\
  \vdots \\
  a_N
\end{bmatrix}
= \begin{bmatrix}
  \tilde{q}_1 \\
  \tilde{q}_2 \\
  \vdots \\
  \tilde{q}_{M_2}
\end{bmatrix} $$

where $u_i^*(P_m) \equiv u_{mi}^*, q_i^*(P_m) \equiv q_{mi}^*, \tilde{u}(P_m) \equiv \tilde{u}_m, \tilde{q}(P_m) = \tilde{q}_m$.

$$ \mathbf{K} \mathbf{a} = \mathbf{f} $$

(10)

The coefficient matrix $\mathbf{K}$ is the matrix of $M \times N$ ($M = M_1 + M_2$). Therefore, if $M = N$, Eq.(10) is solved directly. If $M > N$, it must be solved by applying the least square method.
3 Shape Sensitivity Analysis

We consider the sensitivity analysis with respect to the variation of the shape variable controlling the boundary profile of the object. The potential value \( u(Q) \) and the flux value \( q(Q) \) at the arbitrary point \( Q \) are given by Eqs.(4) and (5), respectively. \( u^* \) and \( q^* \) are dependent on the coordinates of \( Q \) and \( a \) is dependent on the profile of the object and the specified value on the boundary. Therefore, differentiating Eqs.(4) and (5) with respect to the shape variable, we have

\[
\begin{align*}
\dot{u}(Q) &= \dot{a}^T u^*(Q) \\
\dot{q}(Q) &= \dot{a}^T q^*(Q)
\end{align*}
\]

where \( (\cdot) \) denotes the differentiation with respect to the shape variable. \( \dot{a} \) can be calculated from Eq.(10). Differentiating the both sides of Eq.(10), we have

\[
\begin{align*}
Ka + Ka &= \dot{f} \\
Ka &= \dot{f} - \dot{K}a
\end{align*}
\]

If the specified value on the boundary is independent of the shape variable, \( \dot{f} = 0 \) and therefore,

\[
Ka = -\dot{K}a
\]

Equation (13) or (14) is solved for \( \dot{a} \), which is substituted into Eq.(11) in order to calculate the sensitivities. The coefficient matrix of Eq.(14) is the same as that of Eq.(10). Therefore, if \( K \) is LU—decomposed at the initial analysis for calculating \( a \), \( \dot{a} \) can be calculated by the simple operations alone, which is effective for improving the computational efficiency.

4 Numerical Example

4.1 Accuracy verification

The heat transfer in a thick-walled cylinder is considered as a numerical example. Considering the symmetry of the object, the object region for computation is shown in Fig.1. The inner and outer radii and the thickness of the object are \( r_1, r_2 (= 3r_1) \) and \( t (= 3r_1/4) \), respectively. The boundary conditions are specified as follows; \( u = u_2 = 1 \) on the outer wall, \( u = u_1 = 0 \) on the inner wall and \( q = 0 \) on the other walls.

115 collocation points are employed for computation, which are placed uniformly on the whole boundary. The coincident triple collocation points are placed on 8 vertices and the coincident double collocation points are placed on the corner points on the sides of the object. Firstly, we will discuss the convergency of the total errors versus the number of the T-complete
functions. The total errors of the potential and the flux are estimated by the global error estimator defined as

\[
E_u = \frac{1}{M} \sum_{i=1}^{M} (e_u)_i, \\
E_q = \frac{1}{M} \sum_{i=1}^{M} (e_q)_i,
\]
Figure 3: Global errors of potential and flux sensitivities with respect to $r_1$

Figure 4: Distributions of sensitivities with respect to $r_1$

Besides, the local errors at each collocation point are estimated by the local error estimator defined as

$$e_u = \| u - u^{ex} \|$$
$$e_q = \| q - q^{ex} \|$$

where $\| \cdot \|$ denotes the square norm and ($\cdot$) means the value at the collocation point $P_i$. 
Figure 2 indicates the convergency of the global error estimators. The abscissa and ordinate indicate the total number of the T-complete functions and the global error estimators, respectively. The global error estimators uniformly convergence with the increase of the T-complete functions and then, almost zero at $N > 80$. The condition number of the coefficient matrix, which is estimated by the method of Tsukamoto-Natsuka[7], is also indicated in the same figure. Although the condition number is relatively large, the system of equations can be solved accurately.

### 4.2 Shape sensitivity analysis

We consider the sensitivity analysis with respect to variation of the inner radius $r_1$. The global error estimators are defined as

$$E_{su} = \frac{1}{M} \sum_{i=1}^{M} (\varepsilon_{su})_i$$

$$E_{sq} = \frac{1}{M} \sum_{i=1}^{M} (\varepsilon_{sq})_i$$

Besides, the local error estimators are

$$\varepsilon_{su} = ||\dot{u} - \hat{u}^{ex}||$$

$$\varepsilon_{sq} = ||\dot{q} - \hat{q}^{ex}||$$

where $\hat{u}^{ex}$ and $\hat{q}^{ex}$ indicate the theoretical solutions of the potential and the flux sensitivities.

Figure 3 indicates the convergency of the global error estimators versus the total number of the T-complete functions. $E_{su}$ and $E_{sq}$ uniformly convergence with the increase of the T-complete functions. Figure 4 indicates the numerical results at $N = 81$. The abscissa indicates $r$ divided by the inner radius $r_1$ and the left- and right-longitudinal axes the sensitivities of the potential $u$ and its derivative with respect to $r$, $u_r$, respectively. The solid lines and the marks indicate the theoretical and the numerical solutions, respectively. The numerical results well agree with the theoretical ones.

### 5 Conclusions

The Trefftz method is the boundary-type solution procedure formulated by the regular T-complete functions. When the object under consideration is governed by the linear homogeneous differentiation equation, the Trefftz method can solve the problem by taking the collocation points on the boundary or boundary discretization alone. Moreover, the formulation of the Trefftz method is regular and thus, easier than the boundary element
method employing the singular fundamental solution. For encouraging the industrial application of the Trefftz method, in this paper, we firstly developed the program for the Trefftz analysis of the three-dimensional potential problem and then, extended it to the sensitivity analysis. The heat transfer in the thick-walled cylinder was considered as the numerical example. The numerical results well agreed with the theoretical ones. Therefore, it can be concluded that the theoretical validity of the present schemes is proven.

References


