Interaction effects of multiple cracks during fatigue growth
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Abstract

In this paper, the interaction of multiple cracks and their fatigue crack propagation are investigated for a plane elastic problem. The stress intensity factors are calculated by the dual boundary element method with J-integral technique. The real fatigue extension path of cracks is simulated by an incremental linear extension using the maximum principle stress criterion. Examples of geometries with edge and embedded cracks are analysed. It is noticed that the interaction of multiple cracks depends on loading and geometric conditions of structures.

1 Introduction

The multiple crack fatigue problems, especially the interaction and propagation tendency of multiple cracks catch the attention of scientists. One notices that in some case the failure of the structure is caused by the interaction of some cracks nearby and their connection forming a big main crack under fatigue. In another case the multiple cracks may be less dangerous than a single crack.

The numerical technique used in present work is the dual boundary element method (DBEM). The DBEM incorporates two independent boundary integral equations to overcome the degeneracy of equations on surface of cracks. The usual displacement boundary integral equation is applied on usual boundary and one of surfaces of each crack, and the traction boundary integral equation on the another of surfaces of each crack, general mixed mode multiple-crack problem is solved in a single-region formulation. The effective implement of DBEM for a mixed mode crack problem was proposed by Portela, Aliabadi & Rooke [1]. We have achieved a development of application of DBEM to a multiple cracked body and its fatigue growth[2, 3]. The principal intention of this research concerns on the interaction effect of multiple cracks under fatigue.
2 The dual boundary element method

The DBEM is based on dual equations, that are the displacement and traction boundary integral equations. In the absence of body force, the dual boundary integral representation of the displacement components $u_j$ and traction components $t_j$, at a point $x'$ on the boundary $\Gamma$ of a domain $S$ are:

$$c_j(x')u_j(x') = \int_\Gamma U_j(x',x)t_j(x)d\Gamma - \int_\Gamma T_j(x',x)u_j(x)d\Gamma(x) \quad x' \in \Gamma$$  \hspace{1cm} (1)

$$\frac{1}{2}t_j(x') = n_i(x') \int_\Gamma D_{ij}(x',x)t_k(x)d\Gamma(x) - n_i(x') \int_\Gamma S_{ijk}(x',x)u_k(x)d\Gamma(x) \quad x' \in \Gamma$$  \hspace{1cm} (2)

$U_{ij}$ and $T_{ij}$ represent respectively the Flamant displacement and traction of the fundamental solution. $D_{ijk}$ and $S_{ijk}$ denote their derivatives, respectively. $c_{ij}(x')$ is given by $\delta_{ij}/2$ (Kronecker symbol) for a smooth boundary at the point $x'$. The integrals in equations (1) (2) are regular providing $r\neq0$ ($r$ denotes the distance between the points $x'$ and $x$). As the distance $r$ tends zero, the fundamental solutions exhibit singularities: they are weak singularity of order $\ln 1/r$ in $U_{ij}$, a strong singularity of order $1/r$ in $T_{ij}$ and $D_{ijk}$, and a hypersingularity of order $1/r^2$ in $S_{ijk}$. In two later cases the integrals stand for the Cauchy or Hadamard principal-value integrals. Equations (1) and (2) constitute the basis of the DBEM.

The numerical implementation of the principal-value integrals that arise in above dual boundary integral equations can be carried out by the classical method of singularity subtraction. The original improper integral is transformed into the sum of a regular integral and an integral of the singular function. The former can be easily evaluated by method of Gauss, while the latter then evaluated analytically. We use different method in the implementation of the improper integrals. To the outside boundary integral equations, normal rigid body condition is used to calculate strong singular integrals related to $T_{ij}$, singularity subtraction and analytic integration used to weak singular integrals related to $U_{ij}$. To the boundaries of cracks, usual rigid body condition cannot be used due to the degeneracy of lips of cracks. Fortunately, when the curved cracks are usually modelled in piece-wise flat cracks, the improper integrals can be carried out effectively by direct analytic integration.

To satisfy continuity conditions of displacement and its derivative on all nodes for the existence of principal-value integrals, the discontinuous quadratic elements are used as a crack modelling strategy [1]. Crack tips, crack-edge corners and crack kinks do not require special treatment, since they are not located at nodal points where the collocation is carried out. At corner points and the sudden change points of traction applied, double node technique is used. A complete description of the dual boundary element method is given in [1].

The $J$-integral is well used for the determination of stress intensity factors in present method, because the interior elastic field can be accurately determined by using the boundary element formulae after having obtained the solution of boundary. For a mix-mode problem, a simple procedure based on the decomposition of the elastic field into its respective symmetric and
antisymmetric mode components, can be used to decouple the stress intensity factors of a mixed-mode problem[4]. The J integral is represented by the sum of the two integrals as follows:

\[ J = J^I + J^{II} \]  

(3)

where subscript I, II is corresponding respectively to mode I and mode II of crack. To each crack tip, consider a Cartesian reference system, with the origin at the tip of traction-free crack. Now, consider in figure 1 two points \( P(x_1, x_2) \) and \( P'(x_1', x_2') \) symmetric to the crack line relative to the tip. The representation of (3) requires the introduction of the decomposition of displacements and stresses in the elastic fields, see ref. [4].

Then, the J-integral is decomposed into two parts corresponding symmetric and antisymmetric mode components:

\[ J^m = \int r(W^n_n - t^n_n u^n_m) d\Gamma \]  

with \( m \in \{ I, II \} \). Finally, one may have the following relations:

\[ J^I = \frac{K_I^2}{\eta E}, \quad J^{II} = \frac{K_{II}^2}{\eta E} \]  

(5)

In present work, circular paths centred at the tip and containing a pair of crack nodes are used for each crack tip. The integration along the contour path is accomplished by means of the trapezoidal rule. The sign of \( K_{II} \) can be defined by relative displacement solution of crack surface [3].

3 Fatigue crack growth analysis

A structure with multiple edge or centre cracks is considered. In general, the path of crack growth is curved path. In the present approach it is simulated by an incremental crack extension, that is to assume a piece-wise linear discretization of crack path. For each increment analysis, crack extension is conveniently modelled with new boundary elements. In such a way, remeshing is not longer necessary. Among several criteria for the specification of the
direction of crack growth under in-phase mixed-mode loading, we choose the maximum principal stress criterion[5] owing to its efficiency and simplicity, which postulates that the growth of the crack will occur in a direction perpendicular to the maximum principal stress. Thus, at each crack tip the local direction of crack growth $\theta_i$ is determined by the condition that the local shear stress is zero, that is:

$$K_{ii} \sin \theta_i + K_{iii} (3 \cos \theta_i - 1) = 0$$

where $\theta_i$ is the crack growth angle co-ordinate centred at $i$-th crack tip, see also figure 1. The equivalent mode I stress intensity factor is defined at $i$-th crack tip:

$$K_{i\text{eq}} = K_{ii} \cos^3 \frac{\theta_i}{2} - 3 K_{ii} \cos^2 \frac{\theta_i}{2} \sin \frac{\theta_i}{2}$$

The fracture condition then follows from $K_{i\text{eq}} \geq K_{IC}$, in which $K_{i\text{eq}}$ is the maximum value of $K_{i\text{eqi}}$ ($i=1\sim n$, $n$ is number of crack tips), $K_{IC}$ is the fracture toughness of the material. As it is pointed out in [6], crack growth angles defined by this method do not take account of the discreteness of the extension. Hence uniqueness of the crack path cannot be assured with different sizes of the crack-extension increment. Fortunately, the computational and experimental experience shows that the crack initially grows generally in such direction that mode II stress intensity factors tend to vanish. Then, the direction of crack-growth may change slightly. In this case, the influence of $\Delta a$ on predicted path is not very important. On the other hand, if the selected size of $\Delta a$ is too small, some difficulties may arise in calculating the stress intensity factors after the extension. Basing on these consideration, this problem is simply dealt with: one crack tip having maximum value of $K_{i\text{eq}}$ is selected as principal crack tip. And first crack extension $\Delta a$ at principal crack tip is chosen smaller (e.g. two times the length of the initial crack-tip element) than later increment (e.g. three times of length of the initial crack-tip element). The relative extensions at other crack tips, $\Delta a_i$ can be simply evaluated by following relation proportional:

$$\Delta a_i = \Delta a \left( \Delta K_{eff} / \Delta K_{eff} \right)^m$$

This relation is derived from later eqn. (9), in which $m$ is constant of Paris' law. Then the number of boundary elements incremental at each crack-tip is determined according to the size of $\Delta a_i$. In general, this simple method predicts the satisfactory result of crack growth path. For a more complex case where the direction of the service loading is continually changing, or where the local symmetry at the crack tip is upset during crack growth by the geometry of the component, a correct procedure proposed in [6] can be applied to obtain more precise results.

For a multiple crack-tip system, one needs to carry out the fatigue life calculation only at a principal crack tip which can be selected according to values of initial effective stress intensity factor of the crack tips. In order to show the variation in the number of loading cycles as a function of crack length, we take a generalised Paris mode defined as:

$$\frac{da}{dN} = C (\Delta K_{eff})^m$$

(9)
where $a$ is the crack length at principal crack tip selected, $N$ is the number of load cycles, $C$ and $m$ are material dependent constants, and $\Delta K_{\text{eff}}$ is the effective stress intensity factor range. A complete description is given in [3].

4 Results

The two collinear edge cracks in fatigue growth

A rectangular plate ($2H/2W=2$), with a pair of collinear edge cracks of length $a$, is subjected to uniform traction at the two ends. The 32 quadratic elements are used on external boundary, and 6 discontinuous elements on each surface of cracks. The comparison of present results with accurate results of Ref. [7], shows high efficiency of present method, see table 1. On last line of table 1, the result of same geometry but having only one edge crack (for $a/w=0.5$) is also given out. It is noted that stress intensity factor for single edge crack is increased about 28% in comparison with that of two collinear edge cracks in spite of the fact that there is a greater average stress in the mid-section of two edge crack body than that of single edge crack case. The results of fatigue analysis in Figure 2 for initial $a/w=0.3$, show also that the single edge crack grows more rapid and has less fatigue strength and life than the two edge crack structure. It is concluded that the symmetry of two edge cracks decrease the singularity of each crack. Cracks are predicted to propagate along initial linear path in both cases because the stress intensity factors of mode II remain to zero.

Table 1 Stress intensity factors for a pair of collinear edge cracks ($h/w=2$)

<table>
<thead>
<tr>
<th>$a/w$</th>
<th>0.2</th>
<th>0.3</th>
<th>0.4</th>
<th>0.5</th>
<th>0.6</th>
<th>0.7</th>
<th>0.8</th>
<th>Ref.</th>
</tr>
</thead>
<tbody>
<tr>
<td>$K_1/\sigma \sqrt{(\pi a)}$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1.113</td>
<td>1.115</td>
<td>1.132</td>
<td>1.169</td>
<td>1.236</td>
<td>1.353</td>
<td>1.574</td>
<td>[7]</td>
<td></td>
</tr>
<tr>
<td>1.1124</td>
<td>1.1156</td>
<td>1.1334</td>
<td>1.1711</td>
<td>1.2386</td>
<td>1.3567</td>
<td>1.5802</td>
<td>present</td>
<td></td>
</tr>
</tbody>
</table>

| $a$ edge crack | 1.5004 | present |

Fig. 2 Fatigue-life diagram for a pair of collinear edged crack problem
**Two no-collinear internal cracks problem**

A pair of no-collinear cracks of same semi-length \( a \) are in interior of a rectangle \((2H \times 2W, H/a=33.33, W/a=24)\). 20 quadratic elements in outside boundary and 6 elements at each surface of cracks are used. The interaction of two no-collinear cracks is studied in two cases. First, two cracks have a separation as \( 2h=2a \), one is oriented at an angle \( \phi \) to another, see figure 3. As crack 2 is rotated, its crack tips suffer a monotonic decreasing of mode I stress intensity while the magnitude of the mode II first increases, reaching a maximum between 30° and 50°, and then falls to zero. During this process, \( K_I \) of crack 1 increases continually and reach its maximum value as a single crack when crack 2 is perpendicular to crack 1. The present results are very accordant with Ref. [8], but it seems that there is an error definition of tips of crack 1 in [8]. According to sign of \( K_{II} \) of two cracks, the crack tips don't have tendency of merging considering growth of cracks. It is concluded that the single crack perpendicular to the applied tension represents most danger situation in this case.

![Fig. 3 Stress intensity factors for a pair of no-parallel cracks (h/a=1)](image)

![Fig. 4 Stress intensity factors for a pair of parallel cracks (h/a=0.2)](image)
In the second case (see figure 4), we use $b/a$ as a parameter that represents the degree of overlapping of two cracks. The $h/a$ is always remained as 0.2. It is noted that when $b/a$ is about 2, that is two cracks are placed about interlocked, the stress intensity factors attain the maximal value for $K_I$ at crack-tip $A$ (about 1.526), while at external tip of cracks $B$, the value of $K_I$ is about same as that of a crack having a length of $4a$. When two cracks are placed at long distance, the influence of interaction of two cracks decrease. When two cracks are placed side by side, the stress intensity factor $K_I$ are decreased by about 28% comparing single crack due to a shielding effect, that is the relaxation of stress at all crack-tips.

We consider fatigue growth in a special condition, see fig. 6. The cracks are located at vertical distance of 5. The horizontal distance between two internal crack tips ($A$) is 15. The results are given out in case of initial 44 quadratic elements in outside boundary and 6 elements at each surface of cracks. The prediction of crack growth path by present method fig. 5(a) is in good agreement with a experimental result in a similar structure, see fig. 5(b), which is a real path of crack growth recorded through the surface replica of the cracks on the specimen [9]. Although the direction of service loading remains same as fig. 6, the cracks grow still in a curved manner because the local symmetry at the crack tips-$A$ is upset during crack growth by the geometry of the cracked component.

\[
\Delta a \rightarrow 2a_0 \leftarrow \Delta a
\]

Fig. 5(a) Crack growth path prediction for no-collinear central crack problem. (b) Experimental result of crack growth path by the surface replica technique.

Fig. 6 Stress intensity factors at each crack tip vary with crack growth.
Figure 6 shows that the stress intensity factors at crack tips internal (A) and external (B) are initially similar. Then, the stress intensity factors at crack tips A, (where there are greater interaction of two cracks), increase more rapidly then at crack tips B. But when crack tips-A are overlapped, the stress intensity factors tend afterward to decrease because of stress relaxation, while stress intensity factors increase continuously at crack tips B. Correspondingly, cracks grow more rapidly first at internal crack tip-A, finally at external crack tips-B, see figure 7. After then, cracks grow mainly at external tips B as a single crack.

Conclusions

The DBEM offers a very efficient tool to study complex situation of multiple cracks under fatigue. In linear fracture mechanics, the interaction between multiple cracks seems very clarified. An effort concerning the influence of the plasticity in multiple crack problem is significant.

Reference