Abstract

In this paper the elastoplastic dual boundary element method (EPDBEM) is presented for the analysis of elastoplastic fracture mechanics (EPFM) problems. The dual equations of the method are the displacement and the traction boundary integral equations. The use of these dual equations allows the solution of general mixed-mode crack problems with a single-region formulation. The elastoplastic behaviour is modelled through the use of an approximation for the plastic component of the strain tensor on the region expected to yield. This region is discretized with internal quadratic, quadrilateral and/or triangular cells. The possibility of crack-face contact is also considered by directly enforcing compatibility and equilibrium conditions between the surfaces of the crack. Examples of the application to a slant edge-cracked plate, subjected to a tensile load, and to a kinked crack at a hole, at the edge of a plate subjected to reversed bending, are analysed.

Introduction

A single region boundary element analysis cannot be used for non-symmetric problems because the displacement boundary integral equation, when collocated on coincident points on the crack faces, gives rise to a singular system of algebraic equations. The dual boundary element method, as presented by Portela, Aliabadi and Rooke [2], overcomes the need for subregions, suggested by Blandford, Ingraffea and Liggett [1], by the use of two independent boundary integral equations; that is the displacement equation is applied on one of the crack surfaces and the traction equation on the other. Dual boundary element equations have been applied to solve problems in three-dimensional potential theory by Gray [3], and in three-dimensional elastostatics by Gray, Martha and Ingraffea [4]. More recently the dual boundary element method has been extended to a series of problems ranging from 2-D elastodynamics [5], 2-D inverse crack detection [6], and 2-D thermoelasticity [7] to general three-dimensional LEFM problems [8]. The extension of the dual formulation to the analysis of general mixed-mode EPFM problems was presented in detail in Refs. [9] and [10]. The basics of the method together with some further applications are described in this paper.

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The Elastoplastic DBEM

The formulation proposed in Ref. [9], is briefly presented here. The Von Mises yield criterion and the elastoplastic relationship, described by Mendelson [11], between equivalent total strains and increments of equivalent plastic strain due to a given load increment were adopted. In this formulation, the traction boundary integral equation is used together with the displacement boundary integral equation in a way which overcomes the need for subregions in general mixed-mode EPFM problems.

Assuming that the material is homogeneous, the boundary integral representation of the displacement components \( \dot{u}_i \) (where the dot denotes dependence on the load history) for points at the boundary can be expressed as follows,

\[
c_{ij}\dot{u}_j(z) + \int_\Gamma P^*_{ij}(z,x)\dot{u}_j(x)d\Gamma(x) = \int_\Gamma U^*_{ij}(z,x)\dot{p}_j(x)d\Gamma(x) + \int_\Omega \sigma^*_{jkl}(z,x)\varepsilon^P_{jk}(x)d\Omega(x),
\]

where \( \dot{p}_j \) are the boundary tractions; \( \Gamma \) and \( \Omega \) are the boundary and the domain of the body respectively; \( U^*_{ij}, P^*_{ij} \) and \( \sigma^*_{jkl} \) are the weighting fields. These fields, known as the Kelvin fundamental solutions, represent the generalized displacements, tractions and stresses in an infinite, elastic and homogeneous body subjected to unit forces [12]. The plastic strains are represented by the plastic strain tensor \( \varepsilon^P_{jk} \) and \( c_{ij} \) is a constant that depends on the geometry at the collocation point.

The second equation required for the implementation of the dual boundary element method is the traction boundary integral equation. This can be obtained by the differentiation of the displacement boundary integral equation, given in equation (1), followed by the application of Hooke’s law and the definition of tractions:

\[
\dot{p}_i = \sigma_{ij}n_j,
\]

where \( n_j \) denotes the components of the outward normal vector to the boundary. This results in,

\[
\frac{1}{2}\dot{p}_j(z) = n_i(z) \int_\Gamma U^*_{ij}(z,x)\dot{p}_k(x)d\Gamma(x) - n_i(z) \int_\Gamma P^*_{ij}(z,x)\dot{u}_k(x)d\Gamma(x) + n_i(z) \left[ \int_\Omega \sigma^*_{jkl}(z,x)\varepsilon^q_{kl}(x)d\Omega(x) + \frac{1}{2} f_{ij}(\varepsilon^P_{kl}) \right]
\]

where the double star on the kernels \( U^*_{ij,k}, P^*_{ij,k} \) and \( \sigma^*_{ij,k} \) indicates derivatives of the fundamental solutions used in the displacement integral equation (1). The independent term \( f_{ij} \) results from the differentiation of the domain integral in equation (1).

The stresses at internal points are obtained from the following equation:

\[
\dot{\sigma}_{ij}(z) = \int_\Gamma U^*_{ij,k}(z,x)\dot{p}_k(x)d\Gamma(x) - \int_\Gamma P^*_{ij,k}(z,x)\dot{u}_k(x)d\Gamma(x) + \int_\Omega \sigma^*_{ij,k}(z,x)\varepsilon^P_{kl}(x)d\Omega(x) + f_{ij}(\varepsilon^P_{kl}).
\]

Contact Mechanics

The boundary element method is particularly efficient for contact problems since the primary unknowns are all boundary quantities, namely the interfacial tractions
and displacements, and the extent of the contact region. Several different formulations for contact problems have been reported in the BEM literature. The direct constraint method, where the compatibility and equilibrium equations are directly enforced for all the common variables, that is interfacial tractions and displacements, is used here. This is because greater computational efficiency can be achieved when specifically designed assembly procedures and fast matrix solvers are used. This approach to contact problems has been used by Andersson [13], by Karami and Fenner [14] and, more recently, by Man, Aliabadi and Rooke [15]. The contact dependence on factors such as the materials of the two contacting surfaces, their texture and smoothness, can be represented in its simplest form, by a friction law and a friction coefficient.

The modes of contact, that is the different types of boundary conditions, are the separation mode, the slip mode and the stick mode. The equilibrium and compatibility conditions between two nodes $a$ and $b$ on opposite faces of the crack, that must be verified for such states to occur, are described in Table 1: there the tangential and the normal components of the tractions $p$ and the displacements $u$ are denoted by $t$ and $n$ respectively.

<table>
<thead>
<tr>
<th>Separation</th>
<th>Slip</th>
<th>Stick</th>
</tr>
</thead>
<tbody>
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<td>$p_t^a - p_t^b = 0$</td>
<td>$p_t^a - p_t^b = 0$</td>
</tr>
<tr>
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<td>$p_n^a - p_n^b = 0$</td>
<td>$p_n^a - p_n^b = 0$</td>
</tr>
<tr>
<td>$p_t^a = 0$</td>
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</tr>
<tr>
<td>$p_n^a = 0$</td>
<td>$u_n^a + u_n^b = 0$</td>
<td>$u_n^a + u_n^b = 0$</td>
</tr>
</tbody>
</table>

Table 1: Modes of contact.

The boundary modelling consists of discontinuous quadratic elements on the crack surfaces, semi-discontinuous quadratic elements next to the crack mouth and continuous elsewhere; and the domain modelling consists of semi-discontinuous quadratic cells adjacent to the crack and continuous elsewhere. For open traction free cracks, elastoplastic deformation normally occurs at the crack tip only. However if crack closure occurs elastoplastic deformation can occur at other regions on the crack faces. The initial contact conditions are chosen so that the crack faces are in separation mode.

### System Coefficient Matrices

The boundary is discretized into boundary elements and the domain into domain cells. The procedure is described in detail in Ref. [9]. Collocation on all the required boundary and domain points is used to derive the system of linear algebraic equations described in this section.

The boundary equation (1) can be written in the following matrix form:

$$Hu = Gp + Dp.'$$

(5)

The matrices $H$ and $G$ are exactly the same as those obtained for a purely elastic case. Matrix $D$ accounts for the inelastic strain influence over the discretized domain. It should be noted that all the matrices in equation (5) contain terms coming from both displacement and traction boundary integral equations, (1) and (3) respectively. A similar expression can be found for the internal stress equation (4), once it is applied to
all discretized domain points,
\[ \dot{\sigma} = G' \dot{p} - H' \dot{u} + (D' + C') \dot{\varepsilon}^p. \] (6)

The free term in equation (4) is represented in matrix \( C' \) whilst \( D' \) accounts for the inelastic strain influence over the discretized domain.

Reordering equation (5) leads to,
\[ A\dot{y} = \dot{f} + D\dot{\varepsilon}^p \] (7)
where \( \dot{y} \) is the vector of the unknowns, \( \dot{f} \) is the elastic part of the independent term and \( A \) corresponds to the system matrix. Similarly for equation (6),
\[ \dot{\sigma} = -A'\dot{y} + \dot{f}' + E'\dot{\varepsilon}^p, \] (8)
where \( E' = D' + C' \) accounts for the inelastic strains influence and where \( A', \dot{f}' \) and \( \dot{y} \) correspond to the system matrix, the elastic part of the independent term and the vector of the unknowns.

For a traction-free crack, the vector \( \dot{y} \) in equation (7) will have, as unknowns at the crack, the displacements in both directions. As can be seen from the possible modes of contact (Table 1), it is necessary to add to the vector of the unknowns the tractions at nodes belonging to the crack. Augmented versions of the system matrix \( A_a \), the vector of unknowns \( \dot{y}_a \) and the independent term \( \dot{f}_a \) have, therefore, to be created. The submatrix \( G_c \), representing the generalized displacements at the crack due to the fundamental solution, is added to the existing \( A \) system, the original one, together with the contact conditions \( C_u \) and \( C_p \) depending on whether those conditions are expressed in terms of displacements or tractions. This new system of equations is represented schematically in equation (9).

\[
\begin{bmatrix}
A & G_c \\
C_u & C_p \\
\end{bmatrix}
\begin{bmatrix}
\dot{y} \\
\dot{p}_c \\
\end{bmatrix} =
\begin{bmatrix}
\dot{f} \\
0 \\
\end{bmatrix} +
\begin{bmatrix}
D \\
0 \\
\end{bmatrix}
\begin{bmatrix}
\dot{\varepsilon}^p \\
\end{bmatrix}
\]

\[ A_a\dot{y}_a = \dot{f}_a + D_a\dot{\varepsilon}^p \] (9)

In a similar manner matrix \( A' \) in equation (8) is augmented with the corresponding \( G'_c \) submatrix so that the matrix product in equation (8) is consistent. This system is represented in equation (10).
For this type of contact problem the extent of the contact region is not known \textit{a priori} and so must be determined as part of the solution. The initial elastic solution involves the inverse of the system matrix $A_a$; this becomes a very time consuming task if the whole matrix has to be inverted every time the contact conditions change. However, because of the nature of the formulation, boundary condition changes due to changes in contact can be isolated. The matrix equation is structured as above and the inverse can be obtained in a much more efficient way using procedures such as the one described by Hager [16].

### Incremental and Iterative Strategies

Physically non-linear problems are usually solved by adopting a load incremental procedure and by iterating on a particular equation. If the unknowns (initial strains in our case) appear explicitly and are not known \textit{a priori}, a recursive relationship between the stresses, the boundary unknowns and the plastic strains must be used. Iterations are carried out until equilibrium, compatibility and the constitutive/plasticity relationships are all satisfied.

Basically, first the elastic contact mechanics problem is solved to determine both the contact region and the contact conditions. Then elastoplastic behaviour is assumed and a fully elastoplastic analysis is carried out. This procedure is correct for simple elastoplastic contact situations where the contact conditions remain the same throughout the loading process. If these are likely to change then they must be checked at the end of each load increment and updated if necessary.

### Evaluation of J type EPFM parameters

Amongst the several EPFM parameters that can be used, the most versatile are probably the contour integrals known as J integrals. In a recent work by the authors [17], a method was proposed and a study was conducted on the suitability of several of these integrals. For a monotonically applied load of less than say 90% of the load that causes plastic failure, then the well-known Rice's J-integral [18], which for a general contour $\Gamma'$ is given by

$$ J = \int_{\Gamma'} (W_c n_1 - p_i u_{i,1}) d\Gamma + \int_{\Gamma'} W_p n_1 d\Gamma, $$  

is considered to be an appropriate J-integral; it is the one used here.
The proposed formulation has been applied to two problems, a slant edge-cracked plate subjected to a tensile load and a kinked crack at a hole at the edge of a plate subjected to uniform bending.

Slant edge-cracked plate

Consider the analysis of an edge slant crack in a rectangular plate, represented in figure 1. The discretization of the problem consists of a total of 26 boundary elements and 24 internal quadrilateral cells, see Figure 2. A similar problem was analysed by Burstow and Wearing [19] using a combined FEM-BEM method in which the crack region was modelled using finite elements. Here elastic-perfectly-plastic behaviour is assumed, with a yield stress \( \sigma_Y = 100\text{MPa} \); a uniform load of magnitude \( \sigma_n = 0.422\sigma_Y \), was applied at the ends. Plane strain conditions were assumed, with a Young’s modulus \( E = 20000\text{MPa} \) and a Poisson’s ratio \( \nu = 0.3 \). Extrapolated elastic and elastoplastic J values are compared at different load levels in figure 3. This problem cannot be solved by the previous method [17] without partition, since it is not symmetric.

Figure 1: Slant edge-cracked plate: geometry and loading (\( \sigma_n = 0.422\sigma_Y \)).

Figure 2: Slant edge-cracked plate: interior mesh.
Kinked crack from a hole

Karami and Fenner [14], used a purely elastic formulation of BEM to obtain results for an edge-cracked plate under the following bending conditions:

- positive bending, denoted by $M^+$, which causes the crack to open along its whole length, see Figure 4.a;
- negative bending, denoted by $M^-$, which causes the crack to close along some or all of its length, see Figure 4.a.

A range of cracks was considered for which the projected length of the cracks onto the horizontal axis was 0.60, 0.70 and 0.80 of the width of the plate. For negative bending ($M^-$) cracks of length less than or equal to the distance between the crack mouth and the neutral axis of the uncracked plate close completely. However if the crack is long enough to intersect the neutral axis of the uncracked specimen then not all of the surface comes into contact; part of it, just behind the tip, remains open and so there is a non-zero stress intensity factor. This was observed by Paris and Tada [20] and Bowie and Freese [21] who published stress intensity factors for negative and positive bending of an elastic edge-cracked plate and showed that the stress intensity factor, where it exists for $M^-$ is much smaller than that for $M^+$.

Similar results to the ones presented in Refs. [20] and [14] are reproduced here for the case of a kinked crack coming from a hole at the edge of a plate. The geometry is shown in Figure 4.a with the width $W = 0.1m$ and the material properties are as follows: Young’s modulus, $E = 210000\text{MPa}$ (steel), Poisson’s ratio $\nu = 0.3$, yield stress $\sigma_Y = 500\text{MPa}$, plastic modulus $E_T = 0.$ (elastic-perfectly-plastic material). The discretization used is shown in Figure 4.b. The three contours shown in Figure 4.b were used for the calculation of the $J$ integrals. In the subsequent tables and figures the $J$ values presented are the average of the $J$ values for the three contours. Path-independence, within 2%, was verified for all the analyses. A normalized value of the $J$-integral is used for comparison purposes; it is defined by

$$J = \sqrt{\frac{EJ}{a\sigma_Y^2}}.$$  

(12)
Figure 4: Kinked crack from hole: a) geometry and loading; b) discretization used for $a/W = 0.7$. The position marked $x$ is that of the crack-tip and the highlighted lines correspond to the paths for the $J$-integral.
\( J \) has the advantage of being a linear function of the applied load for elastic analysis. Any deviation from linearity is due to elastoplastic behaviour.

In order to compare the fracture parameter \( J \) (or \( J^* \)) for different crack lengths a definition of the loading with respect to some reference stress is required. In general the load that causes fracture when applied to a cracked component decreases with increasing crack length. For tensile loading the average stress at the net section is often used as the reference stress. However, for pure bending problems that is not possible because the average stress at any section is zero. Another alternative is to assume a state of pure bending at the net section and to define the maximum value \( (\sigma_{eq \ max}) \) of the equivalent stress \( \sigma_{eq} \), at the furthest point from the neutral axis, to be related to the yield stress of the material. The relation chosen was \( \sigma_{eq \ max} = 1.75\sigma_y \). The load that it is necessary to apply to the component, so that the maximum equivalent stress takes this value, depends on the length of the crack. The applied load must be expressed in terms of the equivalent stress. The criterion that the bending moment at the net section is the same as the applied bending moment at the ends leads to

\[
\sigma_{eq \ max} \left( \frac{W - a}{2} \right)^{1/6} = \sigma_{app \ max} \left( \frac{W}{2} \right)^{1/6},
\]

which determines \( \sigma_{app \ max} \), the applied stress at the furthest point from the neutral axis of the uncracked section. When \( \mathcal{J} \) is represented as function of the load, the parameter \( \sigma \) is used to express the proportion of the maximum load that had been applied,

\[
\sigma = \frac{\sigma_{current}}{\sigma_{app \ max}}.
\]

The analysis proceeds by first using an elastic model to assess the accuracy of the contact technique developed. The elastoplastic formulation is then applied to the same problem and the differences between the two solutions are shown. Table 2 shows the extent of the contact region \( c/W \) and the normalized \( J \) values for elastic and plastic analysis for positive and negative bending moments using the formulation here described. In this table the following notations are used:

- \( \mathcal{J}_e^+ \) represents the normalized values of the \( J \) integral for the elastic analysis of the plate subjected to positive bending;
- \( \mathcal{J}_e^- \) represents the normalized values of the \( J \) integral for the elastic analysis of the plate subjected to negative bending;
- \( \mathcal{J}_p^+ \) represents the normalized values of the \( J \) integral for the elastoplastic analysis of the plate subjected to positive bending;
- \( \mathcal{J}_p^- \) represents the normalized values of the \( J \) integral for the elastoplastic analysis of the plate subjected to negative bending. When the behaviour is elastic then \( \mathcal{J}_p^- = \mathcal{J}_e^- \).

To highlight the relative importance of the different directions of bending, the ratios between the values of the normalized \( \mathcal{J} \) integral for negative and positive bending are shown. In Figure 6 the variation of the normalized \( J \) integral values is plotted against a normalized load for positive bending.

For the loads used here, the extent of the elastoplastic region around the crack tip for the positive bending \( (M^+) \) varies from approximately 10% to 5% of the crack length for increasing crack sizes. These plastic regions are not very big, which is why the elastoplastic results are not very different from the elastic ones as can be seen from Table 2 and Figure 6. At the edge of the plate ahead of the crack-tip another plastic region, in this case compressive, may develop; this is especially likely for shorter cracks. For the shortest crack, a plastic region is developed of approximately twice the size of the tensile plastic zone at the crack.
Figure 5: Kinked crack from hole: deformed shape for $a/W = 0.8$; a) detail of the crack region for the positive bending; b) detail of the crack region, showing crack-face contact, for the negative bending.
<table>
<thead>
<tr>
<th>$\frac{a}{W}$</th>
<th>$M^+$</th>
<th>$M^-$</th>
<th>$\frac{\varepsilon}{W}$</th>
<th>$\frac{J_+}{J_+}$ (%)</th>
<th>$\frac{J^-}{J^-}$ (%)</th>
</tr>
</thead>
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<td>0.030</td>
<td>0.40</td>
<td>5.21</td>
</tr>
</tbody>
</table>

Table 2: Kinked crack from a hole: results of the analysis.

Figure 6: Kinked crack from hole: $J$ values for positive bending.
Conclusions

A two-dimensional single-region elastoplastic boundary element formulation for the analysis of generalized mixed-mode EPFM problems including the effects of contact between the crack faces has been presented. A system of dual integral equations applied to opposite faces of the crack makes possible the analysis of mixed-mode problems within a single-region formulation. The contact conditions in the potential contact region, i.e. the crack faces, are introduced by direct enforcement of the compatibility and equilibrium conditions. Examples of two different types of crack configuration in plates were presented and analysed.

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References


