Model analysis of plates using the dual reciprocity boundary element method
T.W. Davies & F.A. Moslehy

Department of Mechanical and Aerospace Engineering, University of Central Florida, Orlando, Florida, USA

ABSTRACT

This paper presents a new method for determining the natural frequencies and mode shapes for the free vibration of thin elastic plates using the boundary element and dual reciprocity methods. The solution to the plate's equation of motion is assumed to be a separable form. The problem is further simplified by using the fundamental solution of an infinite plate in the reciprocity theorem. Except for the inertia term, all domain integrals are transformed into boundary integrals using the reciprocity theorem. However, the inertia domain integral is evaluated in terms of the boundary nodes by using the dual reciprocity method. In this method, a set of interior points are selected and the deflection at these points is assumed to be a series of approximating functions. The reciprocity theorem is applied to reduce the domain integrals to a boundary integral. To evaluate the boundary integrals, the displacements and rotations are assumed to vary linearly along the boundary. The boundary integrals are discretized and evaluated numerically. The resulting matrix equations are significantly smaller than the finite element formulation for an equivalent problem. Mode shapes for the free vibration of circular and rectangular plates are obtained and compared with analytical and finite element results.

INTRODUCTION

Classical solutions of the thin plate equation are well documented, e.g. [1]. Applications of the boundary element method (BEM) to the plate problem can be found in [2-5]. The dual reciprocity method (DRM) was first described in [6]. Application of the DRM to static plate bending problems is found in [7].

This paper describes the methodology for solving the vibration problem of a thin plate. The basic plate equations are transformed into the BEM formulation, the domain integral is evaluated by DRM, and all equations are discretized and evaluated numerically to determine the mode shapes for several basic plate shapes and boundary conditions.

BASIC PLATE EQUATION

The governing partial differential equation of motion for a thin plate is
where \( \nabla^4 \) is the biharmonic operator, \( \rho \) is the plate mass density, \( D \) is the flexural rigidity, \( h \) is the thickness, \( q \) is the pressure loading and \( w \) is the lateral or out of plane displacement. For the free vibration problem, the pressure loading term \( q(x, y, t) \) is set to zero and the solution is assumed to be of separable form. Then, for a nontrivial solution, the following equation must be satisfied.

\[
(2) \quad D\nabla^4 w(x, y) - \rho h \omega^2 w(x, y) = 0
\]

In the following analysis, equation (2) will be solved using boundary element and the dual reciprocity methods.

**FUNDAMENTAL SOLUTION**

The exact solution to equation (2) uses Kelvin functions. However, for use in the reciprocity theorem, we formulate a somewhat simpler problem for one of the equilibrium states, an infinite plate subjected to a concentrated load

\[
(3) \quad D\nabla^4 \hat{w}(x, y) = \delta(x, y)
\]

This equation has the solution

\[
(4) \quad \hat{w} = -\frac{r^2}{8\pi D} \ln(r)
\]

This solution is used in the Betti-Rayleigh reciprocal theorem to develop the integral formulation of the BEM for thin plates.

**BOUNDARY ELEMENT FORMULATION**

The integral formulation outlined here follows the methodology developed by Kamiya and Sawaki [7]. After multiplying equation (1) by \( \hat{w} \) and applying the Betti-Rayleigh reciprocal theorem, the following equation results.

\[
(5) \quad \int_\Omega (q \hat{w} - \hat{q} w) d\Omega = D \int_\Omega (\hat{w} \nabla^4 w - w \nabla^4 \hat{w}) d\Omega
\]
Integrating equation (5) reduces the domain integrals to line integrals along the boundary and a remaining domain integral as given by equation (6)

\[ (6) \quad c_i w_i = \int_{\Gamma} \left[ \hat{\omega} K(w) - \frac{\partial \hat{\omega}}{\partial n} M(w) + \frac{\partial w}{\partial n} M(\hat{\omega}) - w K(\hat{\omega}) \right] d\Gamma + \int_{\Omega} \hat{\omega} \rho \omega^2 w d\Omega \]

where \( \Gamma \) is the boundary and \( \Omega \) is the domain.

The constant \( c \) depends on the location of point \( i \); that is, for points in the domain it is one, and for points on the boundary is the included angle between the two elements that join at the node point \( i \). Thus, \( c_i = \theta / 2\pi \). The remaining domain integrals contain the pressure loading term, which is zero for free vibration, and the other integral contains the inertia term. The slope equation is obtained by differentiating equation (6) with respect to the outward normal at the source boundary point. The \( K \) and \( M \) terms in equation (6) are defined below.

\[ (7) \quad K(\ldots) = D \{ (1 - v) \frac{\partial}{\partial s} [n_x n_y \left( \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} \right) - (n_x^2 - n_y^2) \frac{\partial^2}{\partial x \partial y}] - \frac{\partial}{\partial n} \} \{ \ldots \} \]

\[ M(\ldots) = -D[v \nabla^2 + (1 - v)(n_x^2 \frac{\partial^2}{\partial x^2} + n_y^2 \frac{\partial^2}{\partial y^2} + 2n_x n_y \frac{\partial}{\partial x \partial y})] \{ \ldots \} \]

Equation (6) and the slope equation are integral formulation of the boundary element method for the thin plate equation, i.e. equation (2). There are two integrals in these equations, a line integral along the boundary and a domain integral for the inertia term. The DRM for evaluating the domain integral is outlined in the next section. The line or boundary integrals are evaluated numerically.

**EVALUATION OF DOMAIN INTEGRALS**

The DRM is used to evaluate the domain integrals. It uses an approximating function, which produces a boundary integral that only contains known geometric properties and the boundary node displacements. The deflection is assumed to be of the form:

\[ (8) \quad \tilde{w} = \sum_{i=1}^{N+L} f^{i} \alpha^{i} \]
\[ f^{j} \] is an approximating function with unknown coefficients \( \alpha^{j} \) evaluated at the \( N \) boundary points and \( L \) internal points. The approximating function are selected so that

\[ \nabla^{4} w^{j} = f^{j} \]

Then, the domain integral for the inertia term from equation (6) is

\[
\int_{\Omega} \rho \omega^{2} \dot{w} \omega d\Omega = \sum_{j=1}^{N+L} \rho \omega^{2} \alpha^{j} \int_{\Omega} \nabla^{4} w^{j} \dot{w} d\Omega
\]

Applying the Betti-Rayleigh reciprocal theorem to equation (10) results in

\[
\int_{\Omega} \rho \omega^{2} \dot{w} \omega d\Omega = \sum_{j=1}^{N+L} \rho \omega^{2} \alpha^{j} \left\{ c_{j} \dot{w}^{j} + \int_{\Gamma} \left[ \dot{w} \frac{\partial}{\partial n} \nabla^{2} \dot{w}^{j} - \frac{\partial \dot{w}}{\partial n} \nabla^{2} \dot{w}^{j} \right] d\Gamma \right\}
\]

Utilizing the same procedure, a similar expression is obtained for the domain integral in the slope equation. In eqns (11) and the domain integral for the slope equation, the terms in \( \{ \} \) represent known quantities except for \( f \). It is assumed to be of the form

\[ f = C_{0} + C_{1} r + C_{2} r^{2} + C_{3} r^{3} ... \]

The simplest form of the approximating function as suggested by Kamiya and Sawaki, [7], is tested first.

\[ f = 1 + r \]

Additional approximating functions are also tested to improve the accuracy of the DRM in this application. Rewriting equation (8) in matrix form and solving for \( \{ \alpha \} \) yields

\[ \{ \alpha \} = [F]^{-1} \{ w \} \]

Using equation (8) in eqns (11) and (14) and writing in matrix form, we obtain

\[ \{ P \} = \rho \omega^{2} [S][F]^{-1} \{ w \} \]
Equation (15) is the solution for the inertia term in equation (6). These equations are solved in the following section.

**NUMERICAL PROCEDURE**

For the boundary or line integrals in equation (11) and the slope equation, the boundary is divided into elements. A linear interpolating function is used to define the displacement of any point of the linear element in terms of the displacements of the element's end points. Thus, the line integral becomes a function of geometry and material properties only. It is easily evaluated numerically using a Gauss quadrature. The equations are set up in matrix form, rearranged to separate the knowns and the unknowns, and solved to determine all unknown boundary conditions and deflections at the DRM points.

A small computer program has been written to solve equation (6) and the slope equation for the boundary tractions and internal node displacements. The boundary tractions and displacements are rearranged so that the unknown boundary conditions for each required frequency are determined, and the mode shapes are calculated using this program.

After evaluation of all integrals in eqns (11), the equivalent slope equation and the terms of equation (15) are combined and written in matrix form

\[
[H] \begin{bmatrix} w_b \\ \theta_b \\ w_i \end{bmatrix} + [G] \begin{bmatrix} V_b \\ M_b \\ 0 \end{bmatrix} = \rho h \omega^2 [S][F]^{-1} \begin{bmatrix} w_b \\ V_b \\ w_i \end{bmatrix}
\]

In equation (16), the subscript b is for a boundary point and the i is for the internal points. Collecting terms, equation (16) becomes

\[
[H - \rho h \omega^2 SF^{-1}][w] + [G][V] = \{0\}
\]

The vectors \( \{w\} \) and \( \{V\} \) contain both known and unknown boundary conditions and the unknown interior point displacements. Rearranging and collecting all unknown terms onto the right hand side of the equation results in

\[
[A][u] = \{0\}
\]

where \( \{u\} \) is the eigenvector for a particular frequency. In equation (18), for a nontrivial solution for \( \{u\} \), the determinate of \( [A] \) must be zero. Therefore, \( [A] \) can be used to determine the natural frequency of the plate. The value of \( \omega \) is varied and the determinate of \( [A] \) is evaluated to find the zeros of the determinate. The eigenvectors \( \{v\} \) in equation (18) are determined for by separating \( [A] \) into a
lower and an upper triangular matrices and iterating to solve for the eigenvectors. To test the program, several simple plate configurations were analyzed, and the mode shapes determined and compared with FEM calculations.

RESULTS

Several test problems, including a circular and square plates, were solved. Both clamped and simply supported boundary conditions were applied. The material properties for all plates are

\[ E = 200 \text{ GPa} \quad \gamma = 7.68 \text{ kN/m}^3 \quad \nu = 0.3 \]

The circular plate has the following dimensions:

\[ r_o = 254 \text{ mm}; \quad h = 2.54 \text{ mm} \]

The square plate is 254 mm x 254 mm with the same thickness and material properties as the circular plate.

The computed fundamental natural frequencies of the circular plate are 624.9 and 305.6 rad/sec for the clamped and simply supported cases, respectively. The BEM mode shape results are plotted along with the finite element results, run on SAPIV. These results are shown in figures 1 through 6. Figure 1 shows the computed first mode for the simply supported circular plate. The BEM model for the simply supported plate model has 24 boundary nodes, 29 internal points and utilized 4-point Gauss quadrature. The approximating function \( f = 1 + r \) is used in all reported examples, except for the clamped square plate. The FEM model has 125 four-node plate elements and 121 nodes. Figure 2 compares the BEM and FEM results for the axisymmetric second mode for the simply supported plate. Figures 3 and 4 show the results of the first and second mode for the clamped circular plate, respectively. Figure 5 is the first mode results for the simply supported square plate. Figure 6 is the first mode results for the clamped square plate with a revised approximating function \( f = 1 - r - r^2 \). In all cases, the BEM results are in very good agreement with the FEM results.

CONCLUSIONS

As can be seen from the figures in the results section, the BEM and DRM method of calculating the vibration mode shapes for plates agree quite well with the FEM. These results are obtained without discretizing the interior. Interior nodes are described and the inertia contribution of these points is evaluated in terms of the boundary nodes only. This results in matrix equations that are significantly smaller than in the FEM formulation. The simplest form of the
approximating function, equation (13), was used in all cases; no significant
increase in accuracy was obtained with more complicated forms.

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