A simplified boundary element approach for heat transfer in aerospace applications

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ABSTRACT

A boundary element formulation using isoparametric quadratic elements is presented for coupling two-dimensional and axisymmetric zones for steady-state heat transfer applications. The two-dimensional and axisymmetric zones are governed by the modified Helmholtz and Laplace equations, respectively. Turbine jet engines and turbo-machines are two applications where the coupling strategy can be used in the aerospace industry. Rotating machinery or disks are modeled as axisymmetric zones while extended surfaces such as fins or blades are considered two-dimensional. Two examples are presented to demonstrate the accuracy of the proposed coupling technique.

1. INTRODUCTION

Three-dimensional industrial applications require considerable modeling efforts and large computing times [1-3]. Engineers in industry tend to simplify three-dimensional problems by using, whenever possible, two-dimensional and axisymmetric approximations [3,4]. These approximations in many instances do not exactly model three-dimensional effects. However, they do provide the analyst significant insight into the problem under consideration and sometimes very accurate results can be obtained with significantly less modeling effort and computer time [3,4].

Many engineering applications in the aerospace industry, are geometrically very complex, e.g., turbine jet and rotor engines. These structures are composed of different dimensionality components, i.e., two-dimensional components represented by extended surfaces and axisymmetric components represented by the rotating machinery. A steady-state heat transfer analysis of two-dimensional and axisymmetric components are governed by the modified Helmholtz and Laplace equations, respectively.
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In this paper a new coupling technique for two-dimensional and axisymmetric zones [5,6] is presented for the heat transfer analysis of such aerospace structures as opposed to complex and expensive three-dimensional boundary element models. Work presented in [5,6] considered the two-dimensional domain to be governed by the Laplace equation while this study herein considers the modified Helmholtz equation [7,8]. The authors are only aware of one other effort which has coupled two-dimensional and axisymmetric zones for elasticity applications [9]. Isoparametric quadratic elements are used herein to model the boundary of the coupled model, and the multi-zone (substructuring) concept is employed to couple zones by satisfying specific compatibility requirements at their interfacial boundaries. Two numerical examples are presented to demonstrate the accuracy of this coupling technique compared with three-dimensional boundary element models.

2. GOVERNING EQUATIONS & BOUNDARY INTEGRAL EQUATIONS

The Laplace equation, for two-dimensional or axisymmetric, inhomogeneous, isotropic domain comprised of an arbitrary number of homogeneous subdomains (zones) and bounded by a closed surface \( \Gamma \) is given as [10-12]

\[
k_i \nabla^2 u_i = 0 \quad i = 1, \ldots, N \quad \text{in } \Omega_i
\]  

where the subscript \( i \) denotes the zone number and summation is not implied by the repeated subscripts, \( u \) is the temperature, \( N \) is the number of zones and \( k_i \) is the thermal conductivity of zone \( i \).

For two-dimensional zones representing extended surfaces as shown in Figure 1, the modified Helmholtz equation [13] which governs a zone is

\[
\nabla^2 u_i' - a_i u_i' = 0 \quad i = 1, \ldots, N \quad \text{in } \Omega_i
\]

where the positive and known constant parameter for zone \( i \) is

\[
a_i = \frac{h_f + h_b}{kt}
\]

where \( k \) denotes the thermal conductivity, \( t \) is the plate thickness, \( h_f \) and \( h_b \) are the heat transfer coefficients of the forward and backward convecting faces of the plate, respectively, and \( u_f \) and \( u_b \) are the ambient gas temperature at the forward and backward faces of the plate, respectively. The temperature \( u \) is related to \( u' \) through

\[
u' = u - u_s
\]

in which \( u_s \) is given as

\[
u_s = \frac{h_f u_f + h_b u_b}{h_f + h_b}
\]
One should note that a two-dimensional zone governed by the Laplace equation in (1) assumes that the backward and forward faces in Figure 1 are insulated.

Figure 1. Physical parameters for an extended surface.

Boundary conditions considered in this study are of the following three types:

\[
\begin{align*}
    u &= \bar{u} & \text{Dirichlet} & \quad \text{on } \Gamma_1 \\
    q &= -k \frac{\partial u}{\partial n} = \bar{q} & \text{Neumann} & \quad \text{on } \Gamma_2 \\
    \frac{\partial u}{\partial n} &= \frac{h}{k_i} (\bar{u}_f - u) & \text{Robin} & \quad \text{on } \Gamma_3
\end{align*}
\]

where \( \Gamma = \Gamma_1 + \Gamma_2 + \Gamma_3 \) denotes the boundary, \( \bar{u} \) is the prescribed boundary temperature, \( \bar{q} \) is the prescribed boundary flux density, \( n \) is the outward unit normal to the boundary, \( h \) is the heat transfer coefficient and \( \bar{u}_f \) is the ambient fluid temperature.

A general form of the boundary integral equation (BIE) associated with the two-dimensional or axisymmetric zone governed by the Laplace equation in (1) can be stated as \([10-12]\)

\[
c(P)u(P) + \int_{\Gamma} q^*(P,Q) u(Q) R \, d\Gamma(Q) = \int_{\Gamma} u^*(P,Q) q(Q) R \, d\Gamma(Q)
\]

where \( P \) is the source point, \( Q \) is the field point, \( c \) is a free term representing the geometrical interior angle of the boundary at the location of the source point \( P \). Also, \( u^* \) is the boundary temperature, \( q^* \) is the boundary heat flux density, and \( u^* \) and \( q^* \) are the fundamental functions. The function \( R \) is equal to unity for two-dimensional domains and equal to the radial coordinate for axisymmetric domains. The modified Helmholtz equation in (2) has the same integral equation form as the Laplace equation in (7) when \( u \) is replaced by \( u' \). The formulation of
the coupling strategy in the next section will consider the two-dimensional
domain to be governed by the Laplace equation since the BEM equations can be
written directly in terms of temperature u.

The fundamental solution \( u^* \) depends on governing differential equation as
well as on the dimensionality of the domain under consideration. For two-dimen-
sional domains \( 2d \), considering the cartesian coordinates \( x \) and \( y \), the fundamental
solution is \([10-12,13]\)

\[
\begin{cases}
- \frac{1}{2\pi} \ln(r) & \text{for Laplace equation} \\
\frac{1}{2\pi} K_0(\sqrt{\alpha_1} r) & \text{for modified Helmholtz equation}
\end{cases}
\]

where \( K_0 \) is the modified Bessel function of the second kind and zero order and \( r \) is the distance between the source point and the field point. For axisymmetric
domains \( ax \), considering the cylindrical coordinates, \( R \) and \( Z \), the fundamental
solution \([10-12]\) for Laplace equation is

\[
\frac{1}{2\pi^2} K(m,\frac{\pi}{2})
\]

where \( K \) is the complete elliptic integral of the first kind, with \( m \) and \( \frac{\pi}{2} \), being
its module and argument, respectively, and \( cc \) is a geometric parameter.

3. COUPLED BOUNDARY ELEMENT FORMULATION

The standard BEM multi-zone formulation discussed in \([10-12]\) is considered
herein for coupling two-dimensional and axisymmetric zones for heat transfer
applications. Consider a domain composed of two components, i.e., a two-di-
mensional component and axisymmetric one, by multizoning the entire domain
into two zones, i.e., two-dimensional zone \( 2d \) for the two-dimensional compo-
nent, and axisymmetric zone \( ax \) for the axisymmetric com-
ponent as shown in Figure 2,

The matrix equation for the \( 2d \) zone is given by

\[
[H_{2d}] \{ u_{2d} \} = [G_{2d}] \{ q_{2d} \}
\]
and for the \( ax \) zone, the matrix equation has the form

\[
\begin{align*}
[H_{ax}] \{ u_{ax} \} &= \left[ G_{ax} \right] \{ q_{ax} \} \\
[H_{2d}] \{ u_{2d} \} &= \left[ G_{2d} \right] \{ f_{2d} \} \\
\end{align*}
\]

where \( H \) and \( G \) with the related subscripts represent the zone influence matrices. The \( \{ q_{2d} \} \) vector represents the flux per unit thickness of the \( 2d \) zone and the \( \{ q_{ax} \} \) vector represents the flux per unit circumferential length of the \( ax \) zone. Therefore, a transformation from flux into total heat flow is required in order to couple the two different dimensionality zones at their interface. On the other hand, the \( [G] \) matrices should be transformed as well in order to maintain the structure of matrix Equations (10) and (11).

Performing the flux/heat flow transformation, the uncoupled matrix Equations (10) and (11) become

\[
\begin{align*}
[H_{ax}] \{ u_{ax} \} &= \left[ G_{ax} \right] \{ f_{ax} \} \\
[H_{2d}] \{ u_{2d} \} &= \left[ G_{2d} \right] \{ f_{2d} \} \\
\end{align*}
\]

The nodal heat flow \( f_j \) is given as

\[
f_j = \alpha_j q_j
\]

where the subscript \( j \) denotes node number and summation is not implied by the repeated indices. The function \( \alpha_j \) is defined as

\[
\alpha_j = \begin{cases} 
a_j & \text{for interfacial nodes} \\ 1 & \text{otherwise} \end{cases}
\]

The transformed influence matrix is

\[
G_{ij}^t = G_{ij} / \alpha_j
\]

where the term \( a_j \) is the area through which nodal flux \( q_j \) transfers, and is given as

\[
a_j = \begin{cases} 
t & \text{for } 2d \text{ zone} \\ 2\pi R_j / n & \text{for } ax \text{ zone} \end{cases}
\]

where \( t \) is the thickness of the \( 2d \) zone, \( R_j \) is the radial coordinate of interfacial node \( j \), and \( n \) is the number of \( 2d \) zones that are attached to the \( ax \) zone.

The compatibility conditions of temperature and heat flow for coupling the two zones through their interfacial (\( int \)) boundary nodes can be stated for temperature continuity as

\[
\{ u_{2d} \}_{int} = \{ u_{ax} \}_{int}
\]
and for enforced heat flow equilibrium as

\[ \{ f_{2d} \}_{\text{int}} = - \{ f_{ax} \}_{\text{int}} \]  

Equations (12) and (13) can be coupled by fulfilling Equations (18) and (19) which yields the BEM system of mixed algebraic equations for the entire problem in terms of the global influence matrices as

\[ [H] \{ u \} = [G'] \{ f \} \]  

Imposing the appropriate boundary conditions through \( \{ u \} \) and \( \{ f \} \), Equation (20) is rearranged in the form

\[ [A] \{ x \} = \{ b \} \]  

where \([A]\) is a block-banded square matrix representing the entire domain, \( \{ x \} \) is the vector of unknown nodal temperatures and fluxes at external nodes (nodal temperatures and heat flows at interfacial nodes), and \( \{ b \} \) is a known vector whose entries are the sums of products of known temperatures or fluxes and the corresponding coefficients of \( H \) and \( G' \) matrices, respectively. Unknown nodal values can be obtained by solving Equation (21) using Gauss elimination and then recovering the boundary (external and interfacial) nodal heat flux through Equation (14).

Once the boundary values of temperature and flux have been computed at all the nodal points, temperatures and fluxes can be calculated at internal points. The temperature is found by implementing Equation (7) in discretized form with \( c = 1 \) over only the zone within which an internal point lies [10-12]. Fluxes can be found by differentiating Equation (7) in discretized form with respect to the corresponding spatial coordinates [10-12].

4. NUMERICAL EXAMPLES

4.1 Example 1: Benchmark

This example is rather simple since it provides a benchmark for the coupled analysis. The domain under consideration is comprised of a cylinder with 24 blades attached to it. Figure 3 shows the two-dimensional and axisymmetric coupled model with a discretization of 34 isoparametric quadratic elements and different types of boundary conditions are applied at the boundaries of the domain. Side AB of the cylinder is considered to have a constant temperature of 100 °F, and sides AC, BD, CE, EF and FD are considered to have Robin type of boundary conditions with ambient gas temperature \( u_t = 30 \) °F and heat transfer coefficient \( h = 20 \text{ Btu}/(\text{hr} \cdot \text{in}^2 \cdot \text{°F}) \) for sides AC and BD, and \( h = 10 \text{ Btu}/(\text{hr} \cdot \text{in}^2 \cdot \text{°F}) \) for sides CE, EF and FD. Both cylinder and blade have thermal conductivity of \( k = 30 \text{ Btu}/(\text{hr} \cdot \text{in} \cdot \text{°F}) \). The blade’s faces have convection with an ambient gas of temperature \( u_t = 30 \) °F and heat transfer coefficient \( h = 10 \text{ Btu}/(\text{hr} \cdot \text{in}^2 \cdot \text{°F}) \).
°F). Heat transfer in the blade is governed by the modified Helmholtz equation in (2).

\[ \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = \frac{q}{k} \]

\[ T(x, y) = T_0 + \sum_{n=1}^{N} \frac{Q_n}{\sqrt{x^2 + y^2}} \]

Figure 3. Coupled two-dimensional model for Example 1.

A three-dimensional BEM analysis is performed for the same problem in order to verify the results of the coupled two-dimensional model. A 15° segment is used for the three-dimensional analysis due to symmetry. Two meshes are considered, the first mesh as shown in Figure 4a consists of two zones, one for the cylinder and the other for the blade with 148 quadratic quadrilateral elements and 258 nodes.

The temperature distributions along the radial axis with zero z coordinate is shown in Figure 5. Excellent agreement can be observed between the coupled and three-dimensional analyses. Figure 6 shows the radial heat flux through the two zones along the z coordinate axis. Two interesting points can be observed about the flux. Firstly, the heat flux curves from the coupled analysis is discontinuous at the interface location, whereas flux from the three-dimensional analysis is continuous throughout the two zones. This is attributed to the enforcement of
heat flow equilibrium through Equation (19). Heat flow continuity was enforced in the formulation of the coupled analysis instead of normal flux continuity at the interface location where the different thickness zones connect. Flux evaluated from the 2d zone was considered to be the interfacial flux because the actual flux converges to that value at the interface. Secondly, the oscillatory behavior of the heat flux from the three-dimensional analysis is due to the high domain aspect ratio (21.49), i.e., the blade is very thin compared to its other dimensions. Elements on the blade faces are very close together, therefore, integrals that involve the temperature kernels (in the BIE) become more quasi-hypersingular and approach hypersingularity which affects adversely the solution accuracy. In order to improve the results of the three-dimensional model, a finer mesh as shown in Figure 4b is considered with 534 quadratic quadrilateral elements and 737 nodes. Figure 7 shows the comparison of radial heat fluxes between the fine three-dimensional model and the same coupled model discussed above. The heat flux from the three-dimensional analysis converged to the steady behavior as represented by results from the coupled analysis.

![Graph](image)

Figure 5. Radial temperature distribution for Example 1.

![Graph](image)

Figure 6. Radial flux distribution for Example 1, comparison (a).
4.2 Example 2: Aircraft Engine Compressor

This example considers a stage 2 blisk of the aircraft engine compressor shown in Figure 8. The geometric configuration of this example is taken from Pisani and Rencis [4]. The problem under consideration is a geometrically complex disk with 48 blades attached to it, the blades are of uniform thickness $t = 0.0698$ in. Figure 9 shows the coupled model where the disk is modeled using 3 zones (I, II, and IV) while the blade is modeled with one zone (III), 71 isoparametric quadratic elements are used in the model.

Figure 10 shows the three-dimensional model where same number of zones are considered with 180 quadratic quadrilateral and triangular elements and 468 nodes used to cover the surface of the whole model. Thermal conductivity for the disk and blades is $k = 30$ Btu/(hr $\cdot$ in$^2$ $\cdot$ °F). A constant temperature of $u = 200$ °F is applied along the boundary segment ABCDEF. Convection boundary conditions with $u_r = 80$ °F and $h = 10$ Btu/(hr $\cdot$ in$^2$ $\cdot$ °F) along the rest of the boundary. Finally, the blade’s faces have convection with an ambient gas of temperature $u_r = 80$ °F and heat transfer coefficient $h = 10$ Btu/(hr $\cdot$ in$^2$ $\cdot$ °F).
Figure 9. Coupled two-dimensional model for Example 2.

Figure 10. Three-dimensional model for Example 2.
The temperature distribution along the blade boundaries, i.e., the path GHIJ is plotted in Figure 11. The results of the coupled model agree closely with those of the three-dimensional model. The maximum relative percentage error is 3.5%. This example demonstrates the successful implementation of the multi-zone coupling technique for geometrically complex problems with curved and irregular boundaries, as well as, very thin components that are very difficult to deal with using the classical three-dimensional BEM analysis.

![Figure 11. Temperature distribution along blade boundaries GHIJ for Example 2.](image)

5. CONCLUSIONS

In this paper a new coupling technique between two-dimensional and axisymmetric BEM zones was formulated for steady-state heat transfer applications. The coupling technique was implemented in a FORTRAN computer program to assess its accuracy and efficiency. Numerical studies demonstrated that the coupling methodology yields accurate results in analyzing heat transfer in complex structures as opposed to a three-dimensional analysis. The technique, also proved very efficient for structures with irregular boundaries and thin components, since BEM results start to deteriorate when the aspect ratio of the domain under consideration becomes high. In addition, a thin three-dimensional domain analysis yields quasi-strongly singular and quasi-hypersingular integration of the kernels involved in the boundary integral equations. A major advantage of the coupled analysis over the three-dimensional classical analysis is the dimensionality reduction, since the time taken for the modeling phase is substantially reduced. Also, the computer time needed for forming the influence matrices and solving the system equations is significantly less than performing a full three-dimensional analysis.

The accuracy of the heat flux solution obtained using this new approach becomes less as the number of the 2d zones (the blades), that are attached to the ax zone (the main rotating component), is reduced. However, in industrial applications the number of blades range from 30 to 100 which is sufficient to
obtain accurate results. Nevertheless, the temperature solution is not affected by the accuracy of the heat flux as was shown in the numerical implementation.

An important problem that was not addressed is the existence of external boundary conditions (usually convection) on axisymmetric component, i.e., the surface of the axisymmetric component that is not in contact with the two-dimensional components. This convection is of practical consideration and should be included in a complete analysis. A remedy for this problem is to model the axisymmetric component as a three-dimensional zone and then couple the three-dimensional zone with the two-dimensional zone representing the extended surface. This procedure ensures a full modeling of the surrounding boundary conditions and alleviates the accuracy issues associated with three-dimensional thin extended surfaces. One should note that as the number of blades increases the presence of the external boundary conditions on the axisymmetric component will have little effect on the solution away from this surface.

ACKNOWLEDGMENT

The first author was funded by a summer support research grant (SSRG) from the Mechanical Engineering Department at Worcester Polytechnic Institute.

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