Coupling FEM/BEM by the alternative
Schwarz method

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Abstract

In this paper we apply the coupling of the boundary integral method and the finite element method to solve a nonlinear problem in the plane. Specifically the boundary value problem consists of a second order quasilinear problem in divergence form in a bounded inner region and the Laplace equation in the corresponding unbounded exterior region, in addition to appropriate transmission conditions. The method of coupling presented here is based on the alternative Schwarz method. Numerical results are presented in order to show the robustness of the method.

1 Introduction

Many boundary value problems arising in engineering are defined on large or even unbounded domains, for which the traditional numerical methods are unsuitable. Usually a characteristic of these problems is that they are nonlinear or nonhomogeneous only in a small region and on the rest of the domain the problem is linear with constant coefficients.

A procedure coupling finite elements and boundary elements is a suitable numerical method for this kind of problems. The idea of this method is to solve by finite elements in the small domain and solve by the boundary element method in the large or unbounded domain, with suitable transmission conditions. These techniques have been analyzed by many authors in the engineering and in the mathematical literature. For linear problems, see for example, Zienkiewicz, Kelly and Bettess [13], Johnson and Nedelec [11], Costabel [4]-[5], Han [9], and for nonlinear problems Berger, Warnecke and Wendland [2], Costabel and Stephan [6], Gatica and Hsiao [7].
In our work we propose an iterative method to calculate the numerical solution of a nonlinear problem in the plane, coupling the finite element method and the boundary element method by the alternative Schwarz method analyzed in P.L. Lions [12] in the context of domain decomposition methods. Usually in the procedures proposed by the authors above, the global unbounded domain problem is transformed into a problem on a bounded domain but with nonlocal or integral boundary conditions. In the iterative method proposed, we solve two standard independent problems at every step and the matching conditions are automatically fulfilled.

The solution by finite elements of the nonlinear problem is found using the $H^{-1}$ least squares method, used for example by Glowinski [8] for fluid mechanic problems. The solution of the integral equation related with the solution of the exterior Dirichlet problem is constructed, as in Johnson [10], by a variational method.

We present the numerical solution of a model problem. We also calculate the solution of this problem defined on a bounded domain, obtained by our domain decomposition method and compare it with the solution obtained by finite elements on the whole domain.

2 The model problem

Let $\Omega$ be a polygonal domain of $\mathbb{R}^2$, $\Gamma$ its boundary, and let $\Omega'$ be the complement of $\Omega$ in $\mathbb{R}^2$. We consider the following problem

$$
-\nabla \cdot (k(x, \|\nabla u_1\|) \nabla u_1) = f \quad \text{in} \ \Omega,
$$

$$
-\Delta u_2 = 0 \quad \text{in} \ \Omega',
$$

$$
u_1 = u_2 \quad \text{on} \ \Gamma,
$$

$$
k(x, \|\nabla u_1\|) \frac{\partial u_1}{\partial n} = \frac{\partial u_2}{\partial n} \quad \text{on} \ \Gamma,
$$

$$
u_1(x) = O(\|x\|^{-1}) \quad \text{on} \ \Gamma,
$$

$$
\|\nabla u_2(x)\| = O(\|x\|^{-2}) \quad \text{as} \ \|x\| \to \infty,
$$

where $\| \cdot \|$ denote the euclidean norm in $\mathbb{R}^2$, $n$ is the outward normal vector to $\Omega$ defined on $\Gamma$, and $O$ denote the Landau symbol.

Gatica and Hsiao [7] give conditions on $k$ which allow to prove existence and uniqueness of the solution and approximation results.

3 The alternative Schwarz method

Let $\Omega_1$ be a polygonal domain in $\mathbb{R}^2$ such that $\Omega \subset \Omega_1$; we denote its boundary $\Sigma$, see Figure 1.

The Schwarz algorithm associated with the overlapping covering of $\mathbb{R}^2$ given by $\Omega_1$ and $\Omega'$ reads as follows.

Let $u^0$ by an arbitrarily chosen initial iterate. Given the $n^{th}$ iterate, $u^n$, for $n = 0, 1, 2, \cdots$, we compute $u^{n+1}$ in two steps.
We first solve

\[(P1) \begin{cases} -\nabla \cdot \left( k(x, \|\nabla u^{n+1}\|) \nabla u^{n+1} \right) = f & \text{in } \Omega, \\
 u^{n+1} = u^n & \text{on } \Sigma, \end{cases}\]

where we denoted

\[f(x) = \begin{cases} f(x) & \forall x \in \Omega \\
 0 & \forall x \in \Omega - \Omega. \end{cases}\]

and

\[k(x, \|\nabla u\|) = \begin{cases} k(x, \|\nabla u\|) & \forall x \in \Omega \\
 1 & \forall x \in \Omega - \Omega. \end{cases}\]

In the second step we solve the following problem in \(\Omega'\)

\[(P2) \begin{cases} -\Delta u^{n+1} = 0 & \text{in } \Omega', \\
u^{n+1} = u^{n+1} & \text{on } \Gamma, \\
u_2(x) = O(\|x\|^{-1}) & \text{as } \|x\| \to \infty, \\
\|\nabla u_2(x)\| = O(\|x\|^{-2}) & \text{as } \|x\| \to \infty, \end{cases}\]

and define the function

\[u^{n+1}(x) = \begin{cases} u_1^{n+1}(x), & x \in \Omega \\
u_2^{n+1}(x), & x \in \Omega'. \end{cases}\]

## 4 Solution of the nonlinear problem \((P1)\)

For simplicity, we will omit the over index \(n\) in problem \((P1)\). We seek the solution of this problem by the \(H^{-1}\) least squares method proposed by Glowinski [8], which is equivalent to reformulate \((P1)\) like an optimal control problem

\[\begin{align*}
(Q1) \quad & \min \limits_{v \in V} \frac{1}{2} \int_{\Omega_1} \|\nabla \psi(v)\|^2 \, d\Omega_1, \\
\end{align*}\]

where \(V = \{v \in H^1(\Omega_1); \ v = u \text{ on } \Sigma\} \) and \(\psi\) depends on \(v\) through

\[\begin{align*}
(Q2) \quad & \begin{cases} -\Delta \psi(v) = -\nabla \cdot (k(x, \|\nabla v\|) \nabla v) - f & \text{in } \Omega_1 \\
\psi(v) = 0 & \text{on } \Sigma. \end{cases} 
\end{align*}\]
4.1 Numerical approximation of the optimal control problem

Let $\tau_h$ be a standard finite element triangulation of $\Omega_1$ and $V_h = \{v_h \in H^1(\Omega_1); v_h|_T \in P_1, \forall T \in \tau_h; \text{ and } v_h(P) = u(P) \forall P \text{ node } \in \Sigma\}$, where $P_1$ is the space of polynomial functions of degree $\leq 1$. We replace the continuous problem $(Q1)$ by the discrete one $(Q1_h)$

$$(Q1_h) \quad \min_{v_h \in V_h} \frac{1}{2} \int_{\Omega_1} \|\nabla \psi(v_h)\|^2 d\Omega_1,$$

where $\psi(v_h)$ is the solution of the discrete problem associated to the state equation $(Q2)$.

The problem $(Q1_h)$ is solved by the Polak-Ribiere version of the conjugate gradient algorithm. If we denote

$$J(v_h) = \frac{1}{2} \int_{\Omega_1} \|\nabla \psi(v_h)\|^2 d\Omega_1,$$

the algorithm can be formulated in the following steps

**Step 0: Initialization**

- $u_h^0 \in V_h$ given
- Compute $g_h^0 \in H^1_0(\Omega_1)$ solving

\[
\begin{cases}
-\Delta g_h^0 = J'(u_h^0) & \text{in } \Omega_1, \\
g_h^0 = 0 & \text{on } \Sigma.
\end{cases}
\]

- $z_h^0 = g_h^0$

Then for $m \geq 0$ assuming $u_h^m$, $g_h^m$, $z_h^m$ known, compute $u_h^{m+1}$, $z_h^{m+1}$, $z_h^{m+1}$ by

**Step 1: Descent**

- Compute $\lambda^m$ such that

$$J(u_h^m - \lambda^m z_h^m) \leq J(u_h^m - \lambda z_h^m) \quad \forall \lambda \in R.$$

- $u_h^{m+1} = u_h^m - \lambda^m z_h^m$.

- If $|J(u_h^{m+1})| < \varepsilon$ Stop. Else

**Step 2: New descent direction**
Define $g_h^{m+1} = H^0(\Omega_1)$ by
\[
\begin{cases}
-\Delta g_h^{m+1} = J'(u_h^{m+1}) & \text{in } \Omega_1, \\
g_h^{m+1} = 0 & \text{on } \Sigma.
\end{cases}
\]

Define
\[
\gamma^m = \frac{\int_{\Omega_1} \nabla g_h^{m+1} \cdot \nabla (g_h^{m+1} - g_h^m) d\Omega_1}{\int_{\Omega_1} \|\nabla g_h^m\|^2 d\Omega_1}.
\]

- $z_h^{m+1} = g_h^{m+1} + \gamma^m z_h^m$.
- $m = m + 1$ and go to Step 1.

The hardest steps of the algorithm are the calculation of $\lambda^m$ and the calculation of the gradient $g_h^{m+1}$. We describe this calculations in some details.

**Computation of $\lambda^m$:** The calculation of $\lambda^m$ by minimizing the function $J(u_h^m - \lambda z_h^m)$ is computationally very expensive because at every step of the algorithm it is necessary to do several evaluations of that function, and at every evaluation we need to solve one Dirichlet problem. For that reason we approximate the function $J(u_h^m - \lambda z_h^m)$ by a quadratic function in $\lambda$, whose exact minimum is given by
\[
\lambda^m = -\frac{\int_{\Omega_1} \nabla \psi(u_h^m) \cdot \nabla \psi_1(u_h^m) d\Omega_1}{\int_{\Omega_1} \|\nabla \psi_1(u_h^m)\|^2 d\Omega_1},
\]
where $\psi(u_h^m)$ and $\psi_1(u_h^m)$ are the solutions two linear Dirichlet problems.

**Computation of $g_h^{m+1}$:** The gradient $g_h^{m+1}$ is calculated by solving the variational problem
\[
\int_{\Omega_1} \nabla g_h^{m+1} \cdot \nabla w_h d\Omega_1 = < J'(u_h^{m+1}), w_h >_{H^{-1}(\Omega_1) \times H^1_0(\Omega_1)}, \quad \forall w_h \in H^1_0(\Omega_1),
\]
and, in our particular case, the right hand side is given by
\[
< J'(u_h^{m+1}), w_h > = \int_{\Omega_1} \tilde{k}(x, \|\nabla u_h^{m+1}\|) \nabla w_h \nabla \psi(u_h^{m+1}) d\Omega_1
\]
\[
+ \int_{\Omega_1} \tilde{k}'(u_h^{m+1}; w_h) \nabla u_h^{m+1} \nabla \psi(u_h^{m+1}) d\Omega_1,
\]
where $k'(u_h^{m+1}; w_h)$ is the Gâteaux derivative of $k$ at the point $u_h^{m+1}$ in the direction $w_h$.

## 5 Resolution of an exterior Dirichlet problem by the BEM

The solution of the exterior Dirichlet problem (P2) satisfies the following integral representation formula [see Atkinson [1]]
\[
\int_\Gamma \left[ \frac{\partial u_2(y)}{\partial n_y} \ln \left( \frac{1}{\|x - y\|} \right) - u_2(y) \frac{\partial}{\partial n_y} \left( \ln \frac{1}{\|x - y\|} \right) \right] d\Gamma_y = \alpha u_2(x), \quad (1)
\]
where

\[
\alpha = \begin{cases} 
0 & x \in \Omega, \\
-\left(2\pi - \theta(x)\right) & x \in \Gamma, \\
-2\pi & x \in \Omega',
\end{cases}
\]

where \( \partial \overline{\partial n_y} \) indicates differentiation in the direction \( n_y \), \( \theta(x) \) denote the solid angle at \( x \in \Gamma \).

Relation (1), with \( x \in \Gamma \), permits the calculation of \( \partial_{n_y}u_2 \) by solving the Fredholm integral equation of the first kind

\[
\int_{\Gamma} \frac{\partial u_2(y)}{\partial n_y} (y) \ln \left( \frac{1}{\|x - y\|} \right) d\Gamma_y = -[2\pi - \theta(x)]u(x) + \int_{\Gamma} u(x) \frac{\partial}{\partial n_y} \left( \ln \frac{1}{\|x - y\|} \right) d\Gamma_y,
\]

where \( u(x) \) is the boundary condition on \( \Gamma \).

The relation (1), with \( x \in \Omega' \), permits the calculation of \( u_2(x) \), where \( \partial_{n_y}u_2 \) is the solution of the last integral equation.

### 5.1 Numerical method for the Fredholm integral equation

There is a wide variety of numerical methods to solve boundary integral equations. Most of them are of collocation type, Brebbia and Dominguez [3], or of Galerkin type, Johnson and Nedelec [11]. We use here the second approach.

In the sequel we write our integral equation in the form

\[
\int_{\Gamma} q(y) \ln \left( \frac{1}{\|x - y\|} \right) d\Gamma_y = G(x),
\]

where

\[
G(x) = -[2\pi - \theta(x)]u(x) + \int_{\Gamma} u(x) \frac{\partial}{\partial n_y} \left( \ln \frac{1}{\|x - y\|} \right) d\Gamma_y
\]

and

\[
q(y) = \frac{\partial u_2(y)}{\partial n_y}(y).
\]

We introduce the Sobolev space

\[
W = \left\{ \mu \in H^{-1/2}(\Gamma); \int_{\Gamma} \mu(x)d\Gamma_x = 0 \right\},
\]

see [11] and the references therein for the definition of the space \( H^{-1/2}(\Gamma) \). The variational formulation of (2) is:

\[
\begin{cases}
\text{Find } q \in W; \\
b(q, p) = l(p) & \forall p \in W,
\end{cases}
\]

(3)
where
\[ b(q, p) = \int_{\Gamma} \int_{\Gamma} \ln \left( \frac{1}{\|x - y\|} \right) q(y)p(x) d\Gamma_y d\Gamma_x, \]
\[ l(p) = \int_{\Gamma} G(x)p(x) d\Gamma_x. \]

Let \( \{K_i\}_{i=1}^N \) be the subdivision of \( \Gamma \) induced by the triangulation \( \tau_h \), and
\[ W_h = \{v_h \in W ; \quad v_h|_{K_i} \text{ constant}, \quad i = 1, 2, \ldots N\}. \]

The discrete version of (3) is:
\[
\begin{cases}
\text{Find } q_h \in W_h; \\
b(q_h, p_h) = l(p_h) \quad \forall p_h \in W_h.
\end{cases}
\]

Problem (3) can be formulated in the following matrix form
\[ BY = C, \]
where \( Y = (y_1, y_2, \ldots y_n)^T \), \( B = (b_{ij}) \), \( C = (c_i) \) with
\[ b_{ij} = \int_{K_i} \int_{K_j} \ln \left( \frac{1}{\|x - y\|} \right) d\Gamma_y d\Gamma_x, \quad i, j = 1, 2, \ldots N, \]
\[ c_i = \int_{K_i} G(x) d\Gamma_x \quad i = 1, 2, \ldots N. \]

A large part of the computational effort is spent computing the coefficients \( b_{ij} \) and \( c_i \) defined by the above integrals.

We must consider two different cases in the calculation of \( b_{ij} \):

1. When the boundary elements \( K_i \) and \( K_j \) are disjoint or have one common point, a Gauss numerical integration formula is used.

2. If \( K_i = K_j \) the integrand contain singularities but \( b_{ii} \) can be evaluated analytically.

Similar considerations permit the calculation of the data \( c_i \), but there is no problem when \( K_i = K_j \).
Table 1: Number of iterations needed to satisfy stopping criteria (5) for a fixed overlapping and different triangulations.

<table>
<thead>
<tr>
<th>h</th>
<th>1/8</th>
<th>1/16</th>
<th>1/32</th>
<th>1/64</th>
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<tbody>
<tr>
<td>NIT</td>
<td>22</td>
<td>16</td>
<td>16</td>
<td>16</td>
</tr>
</tbody>
</table>

Table 2: Number of iterations to satisfy criteria (5) for different sizes of the overlapping.

<table>
<thead>
<tr>
<th>d</th>
<th>1/2</th>
<th>1/4</th>
<th>1/8</th>
<th>1/16</th>
<th>1/32</th>
</tr>
</thead>
<tbody>
<tr>
<td>NIT</td>
<td>11</td>
<td>16</td>
<td>43</td>
<td>88</td>
<td>172</td>
</tr>
</tbody>
</table>

6 Numerical results

We describe some numerical results obtained using the method discussed before to solve the model problem where $\Omega$ is chosen to be the unit square $(0,1) \times (0,1)$ with the following data

$$k(x, \| \nabla u \|) = 2 + \frac{1}{1 + \| \nabla u \|}$$

and

$$f(x) = \sin(\pi x_1) \sin(\pi x_2).$$

Table 1 indicates the number of iterations (NIT) required to satisfy the following stopping criteria

$$\frac{\| u^{n+1} - u^n \|_{\Gamma}}{\| u^n \|_{\Gamma}} \leq 10^{-6},$$

for a fixed overlapping $d = \text{distance}(\Gamma, \Sigma)$, taken equal to 1/4. We have denoted $\| \cdot \|_{\Gamma}$ the euclidean norm of the vector given by the values of the function at the nodes on $\Gamma$. The table indicates that the convergence does not deteriorate when the diameter of the triangulation $h$ becomes small.

In table 2 we can see the influence of the size of the overlapping $d$ on the convergence speed. In this test we have chosen $h = 1/32$.

Finally, in order to have a qualitative idea of the performance of the method, we consider the model problem with the data given above posed in the bounded region $(0,2) \times (0,1)$. We take $\Omega = (0,1) \times (0,1)$ and $\Omega' = (1,2) \times (0,1)$. First, we solved the global problem with the finite element method and then compare the solution with the one obtained by the decomposition method at the fourth step as depicted in figure 2.
Figure 2: The solution obtained by FEM and by decomposition method after 4 iterations.

References


