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A new model for connections between skeletal and continuum structural elements

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Abstract

This paper presents a new approach for the analysis of structures consist of continuum elements coupled with skeletal elements. The proposed approach is based on the analysis of continuum elements using the boundary element method (BEM), whereas the analysis of the skeletal elements is carried out using the traditional flexibility/stiffness method. The assembly of the structural parts is carried out using the well known FEM assembly technique. Some different modeling adjustments were introduced to make such traditional concepts compatible with the boundary element method. The coupling between a skeletal frame with a 2D continuum structure is presented as an application to demonstrate the new concept. The obtained results are compared to those obtained using the finite element method to demonstrate the accuracy and the validity of the newly presented approach.

1 Introduction

Skeletal elements and continuum structures are widely used in buildings and in structures in general. Many structures are designed to include coupled skeletal elements with continuums such as building floors or shear wall systems. The analysis of such structures is done traditionally using finite element (FEM) packages which require the domain to be meshed.

After the boundary element (BEM) method has been emerged, some new approaches were developed to couple both the boundary element and the finite element methods. Coupling both numerical methods could be useful in many engineering applications. One of them is the analysis of coupled skeletal elements with continuums as it will be presented in this paper.

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Several techniques were developed to merge both BEM and FEM methods either by treating the BEM formulation as a special case of FEM formulation, forcing the BEM matrices to be symmetric using the energy minimization method and presenting the BEM matrices as stiffness matrices or to transform the FEM forces into tractions and linking them with the tractions in BEM matrices.

An early work on merging of BEM and FEM was started by Zeinkiewicz et al [1] followed by Shaw and Fably [2]. Many other publications dealt with the same theme were published by Kelly et al [3], Brebbia and Georgiou [4], Margulies [5]. Stein and Kreienmeyer [6] and Elleithy and Al-Gahtani [7] utilized the overlapping domain decomposition method for coupling the BE and FE methods. Recently, another approach was developed by Rashed [8] for coupling BEM-flexibility force method for bending analysis of internally supported plates.

In this paper, a new approach of coupling skeletal frames with 2D continuums using the boundary element method is presented. This approach is based on flexibility/stiffness analysis in which the structure is divided into a series of individual sub-structures. For each substructure the stiffness matrix and load vectors are setup. The overall stiffness matrix and overall load vector is assembled using the well known assembly technique [9]. The obtained system of linear algebraic equations is then solved for the unknown displacements. This procedure is automated using a developed software. As an advantage of the proposed procedure is that it can be introduced easily among engineers because the core of this method is the traditional flexibility/stiffness analysis which is one of the well known methods that has been used in engineering education. A 2D continuum coupled with skeletal element is presented as an application for coupling continuum structures with skeletal elements for two dimensional elasticity. The results obtained are compared to those obtained from solving the same problem using the finite element method.

2 Flexibility and stiffness methods

Flexibility and stiffness methods are well known methods used to solve skeletal structural problems. Flexibility is the displacement due to a unit load in the direction of a degree of freedom [9]. This definition can be written in a matrix form as follows:

$$\{\delta_i\} = [C_{ij}]\square\{F_j\} \tag{1}$$

where $\{\delta_i\}$ is the displacement at node i, $[C_{ij}]$ is the influence coefficient at node i due to a force at j, and $\{F_j\}$ is the force applied at node j. Also, the set of forces $\{F_j\}$ can be obtained in terms of the displacements as follows:

$$\{F_j\} = [C_{ij}]^{-1}\{\delta_i\} \tag{2}$$

where

$$[C_{ij}]^{-1} = [K_{ij}] \tag{3}$$

Equation (2) can be rewritten as follows:

$$\{F_j\} = [K_{ij}]\{\delta_i\} \tag{4}$$

In which $[K_{ij}]$ is the force required at node i to sustain a displacement at j or the stiffness matrix.

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3 The Boundary Element Method

The Boundary Element Method (BEM) is one of the powerful numerical methods in solving engineering problems involving continuum elements. The main advantage of the boundary element method is that the discretisation is done only for the boundary or the surface. This reduces the time of computations and effort of modeling. In this section, the BEM method is reviewed for 2D elasticity problems. The boundary integral equation can be written after neglecting the body forces as follows [10]:

$$C_{ij}(\zeta)u_i(\zeta) + \int_\Gamma T_{ij}(\zeta, x)u_j(x)d\Gamma(x) = \int_\Gamma U_{ij}(\zeta, x)t_j(x)d\Gamma(x) \tag{5}$$

Where C_{ij} is the free term depends on the geometry and equal to 0.5 if the collocation point (ζ) is on a smoothed boundary or 1 if (ζ) is inside the domain, $T_{ij}(\zeta,x)$, $U_{ij}(\zeta,x)$ are the fundamental solution kernels of the tractions and displacements respectively [10] and $u_j(x)$, $t_j(x)$ are the boundary displacement and traction variables at the point (x) in the j direction [10].
The corresponding matrix form for equation (5) can be written as follows [8]:

$$[H]_{3N\times3N}\{u^{(1)}\}_{3N\times1} = [G]_{3N\times9NE}\{t^{(1)}\}_{9NE\times1} \tag{6}$$

where [H] and [G] are the well known influence matrices [10], N and NE denote the number of boundary nodes and elements respectively and {u} and {t} are the vectors of the boundary displacements and tractions respectively. Equation (6) represents the solution due to the original load. A similar set of equations can be rewritten for the solution of the different number of load cases n as follows [8]:

$$[H]_{3N\times3N}[u^{(1)}u^{(2)}...u^{(n)}]_{3N\times n} = [G]_{3N\times9NE}[t^{(1)}t^{(2)}...t^{(n)}]_{9NE\times n} \tag{7}$$

Equation (6) and equation (7) are both valid for the quadratic element.

4 Coupling skeletal elements with continuums

Modeling the connection between skeletal and continuum elements is dependent on the number and the directions of the degrees of freedom which are chosen by the engineer or the modeler. Such degrees of freedom can be either translational or rotational. In this paper, only the translational degrees of freedom will be considered. In order to demonstrate the new developed approach, consider the structure shown in Fig-1 which consists of two continuum elements C_1 and C_2 coupled with a skeletal frame S_1. Two translational degrees of freedom at each connection are considered (at A and B respectively) as shown in Fig-1. Without loosing the generality, here is presented the methodology of solving for coupling skeletal elements with 2D continuums as follows:

1-The degrees of freedom at the connections points A and B respectively are determined (see Fig-2) then, the structure is divided into a series of sub-structural elements as shown in Fig-3.

2-The flexibility matrix and the load vector of each sub-structure are computed. For the continuum elements C_1 and C_2, the flexibility matrix can be obtained using the BEM where each column of the matrix can be computed using different scheme of load cases. Therefore a set of unit loads are applied in the direction of the degrees of freedom. Each load is considered as a different load case as shown in Fig-4 and Fig-5 respectively, these load cases can be considered simultaneously as in equation (7). In the same manner, the flexibility matrix of the skeletal frame S_1 is obtained using FEM by applying a set of unit loads as shown in Fig-6. As an alternative way, the flexibility or stiffness matrices of skeletal elements can be obtained directly using well known techniques [9]. Assume that the flexibility matrices $[C_1]$, $[C_2]$ and $[S_1]$ are the flexibility matrices of the continuum elements C_1, C_2 and the skeletal element S_1 respectively. Then the stiffness matrices $[K_{(\bullet)}]$ can be computed as follows:

$$[K_{(\bullet)}] = [C_{(\bullet)}]^{-1} \tag{8}$$

The load vector $\{L_{(\bullet)}\}$ of each of the continuum elements and the skeletal element can be obtained as follows [9]:

$$\{L_{(\bullet)}\} = -[K_{(\bullet)}]\{\delta_{(\bullet)}\} \tag{9}$$

where $(\bullet)$ can be C_1, C_2 or S_1 and $\{\delta_{(\bullet)}\}$ is the displacement under the original load. It has to be noted that the load in case of the boundary element method is no longer concentrated, it is distributed over the element length as it will be discussed in the next section.

3-The overall stiffness matrix and the corresponding overall load vector are formed using the well known assembly technique [9].

4-The overall solution can be obtained for the unknown displacements by solving a system of linear algebraic equations as shown below:

$$\{\delta\}_{overall} = -[K]^{-1}_{overall}\{L\}_{overall} \tag{10}$$

where, $\{\delta\}_{overall}$ is the displacement vector, $[K]_{overall}$ is the overall stiffness matrix and $\{L\}_{overall}$ is the overall load vector.

5-The post-processing stage comes at the end after obtaining a solution for the unknowns where the final solution of the problem can be written as follows [8]:

$$(\bullet)^{(f)} = (\bullet)^{(1)} + \sum_{k=1}^{n} (\bullet)^{(k)} \times U_k \tag{11}$$

where $(\bullet)$ can be the boundary or internal generalized displacement or traction. The subscript (f) denotes the quantities of the final solution, (1) denotes the initial or the original load case, (k) stands for the number of the load cases and U_k is the unknown displacement.

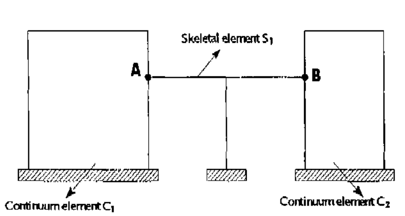

Fig. 1: A coupled skeletal frame with two continuum elements

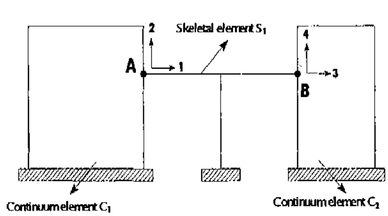

Fig. 2: The proposed degrees of freedom at the Connections points

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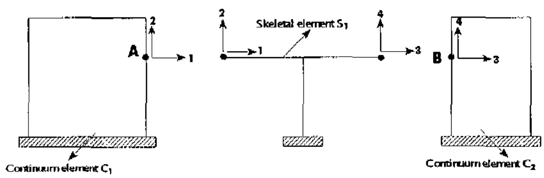


Fig. 3: The proposed sub-structures.

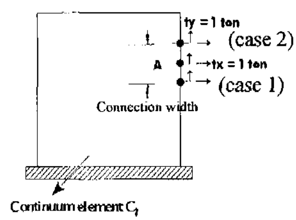


Fig. 4: The element C_1 load cases.

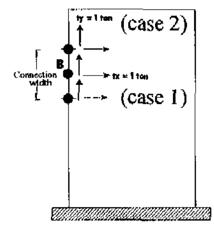


Fig. 5: The element C_2 load cases.

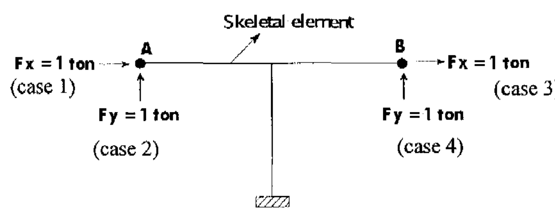


Fig. 6: The skeletal element S_1 load cases.

5 Application: A coupled wall with a skeletal element

In this problem a wall is coupled with a frame having the following material properties: Youngs modulus, $E_{wall}=E_{frame}$ = 1E6 t/m^2 and Poisson's ratio $\mu_{wall} = \mu_{frame} = 0.25$. The frame cross section is 0.4 m length and 0.4 m width. Only two translational degrees of freedom are considered at the connection point A (see Fig. 7), U is horizontal displacement in X-direction and V is the vertical displacement in Y-direction as shown in Fig. 7.

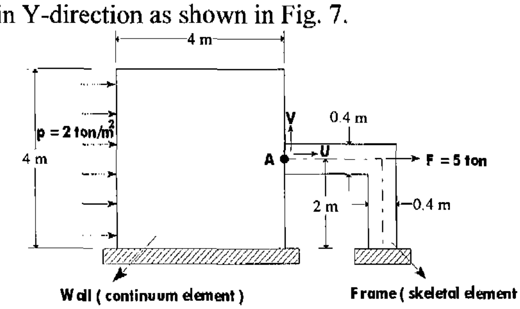


Fig. 7: Coupled wall-frame structure.

5.1 Problem sub-structuring

The model shown in Fig. 7 will be divided into two individual sub-structures, one is a continuum element and the other one is a skeletal element as shown in Fig. 8.

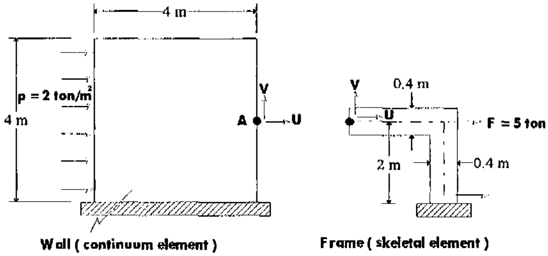


Fig. 8: Sub-structures.

These sub-structures will be analyzed separately. The continuum element or the wall will be analyzed using the boundary element method and the skeletal frame will be analyzed using the well known stiffness analysis.

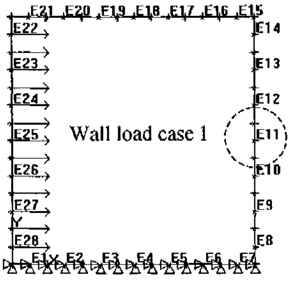
5.2 BEM stiffness model

In this model, distributed and virtually concentrated tractions (as it will be discussed in section 5.5) are used as unit loads to represent the different load cases of the wall (as it will be discussed in section 5.5) in order to generate the flexibility matrix of the wall as shown in Fig. 9 to 11 respectively. The width of the connection between the frame and the wall is considered as one boundary element to simulate the real width of the connection.

In this problem, there are three load cases considered for both of the wall and the skeletal frame as follows:

The Wall load cases:

The wall is discretized into 7 quadratic elements on each side as shown in Fig-9 to 11 respectively. The first wall load case is that when the wall is subjected to the original loading shown in Fig. 9 where a 2 ton/m^2 pressure is applied. In order to generate the wall flexibility matrix, other two wall load cases are applied in which a set of unit tractions is applied at the connection point A in X-direction and Y-direction respectively (see Fig.10 to 11).



The element that demonstrates the connection between the wall and the frame. The length of this element is 0.4 m which is the same as the frame depth.

Fig. 9: The original load.

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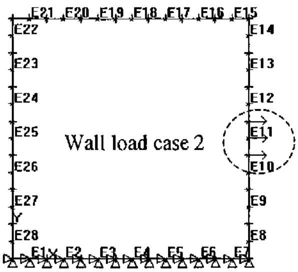
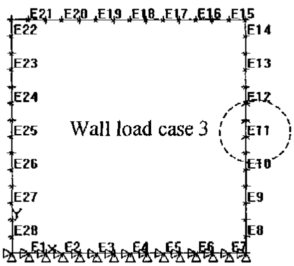


Unit traction is applied in the X-direction which is the direction of the first degree of freedom at the connection between the wall and the frame.

Fig. 10: Generating the first column in the wall flexibility matrix



Unit traction is applied in the Y-direction which is the direction of the second degree of freedom at the connection between the wall and the frame.

Fig. 11: Generating the second in the wall flexibility matrix.

It has to be noted that the first load case marked with the subscript (1) is used to compute the wall load vector according to equations (8) to (10) respectively.

The Skeletal frame load cases

The stiffness matrix of the skeletal element can be obtained and incorporated directly in the solution procedure. However, herein a similar technique via load cases is used to generate it. Such technique is more suitable for using the present model via coupling different ready made software packages where the relevant condensed stiffness matrix is not explicitly printed out.

Three load cases are considered for the frame as shown in Fig-12. These three load cases are to simulate the original loading of the frame and to generate the frame flexibility matrix.

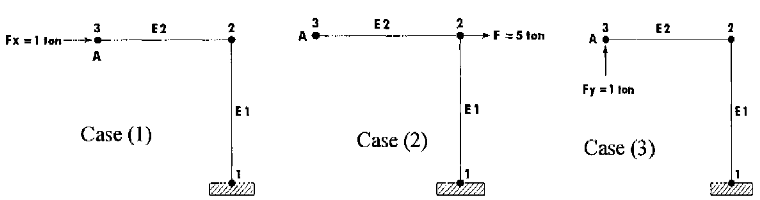


Fig. 12: The frame load cases.

5.3 Assembly of sub-structures

Once the flexibility matrix of each substructure is obtained, the stiffness matrix and the load vector of each sub-structure can be computed, (recall equations (8) to (9)). An overall stiffness matrix and load vector can be obtained by assembling the stiffness matrices and the load vectors of the substructures in one overall stiffness matrix and load vector [9].

5.4 Final solution

The solution for the unknown displacements can be obtained by substitution in equation (10).

5.5 Load models and displacement measurements

In the former computation, the target was to compute the flexibility influence coefficients. To do that, two different aspects has to be considered: (1) load modeling and (2) displacements measurements:

(1) Load modeling:
In the boundary element method, the unit loads are applied as tractions. These tractions can take one of two forms:
i) The tractions are distributed over the element length as shown in Fig-13(a).
ii) The tractions are simulated as virtually concentrated tractions as shown in Fig-13(b).

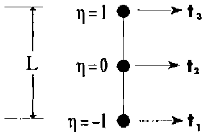


Fig. 13(a): Distributed tractions.

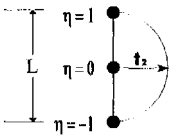


Fig. 13(b): virtually concentrated tractions.

The total force can be expressed according to the load models shown above as follows [11]:
From Fig-13(a) the total force can be computed as follows:

$$\text{Total force (F)} = L(\frac{1}{6}t_1 + \frac{2}{3}t_2 + \frac{1}{6}t_3) = Lt \tag{12}$$

From Fig-13(b), the total force can be computed as follows:

$$\text{Total force (F)} = \frac{2L}{3}t_2 \tag{13}$$

Where L is the element length, t_1, t_2 and t_3 are the tractions at the element nodes respectively in which t_1=t_2=t_3 =t and η is the natural coordinate system.

(2) Displacement measurement:
In order to generate the flexibility matrix of the wall using the boundary element method, the coefficients of this matrix which are the displacements due to the unit loads can be calculated in three different ways as follows:
i)The flexibility matrix can be obtained by measuring the displacements at the middle node of the element.
ii) The flexibility matrix can be obtained by computing the average displacements of the three nodes of the element respectively as follows:

$$u = \frac{u_1 + u_2 + u_3}{3} \tag{14}$$

iii)The flexibility matrix can be obtained by computing the average continuous displacements as in equation (15) and equation (16) respectively.

$$u = \frac{1}{L}\int_{\Gamma} u_1\phi_1 + u_2\phi_2 + u_3\phi_3 d\Gamma \tag{15}$$

Equation (15) can be rewritten analytically as follows [11]:

$$u = \frac{1}{6}u_1 + \frac{2}{3}u_2 + \frac{1}{6}u_3 \tag{16}$$

where L is the element length, u_1, u_2 and u_3 are the displacements of the element nodes respectively and $\phi_{\square}$, ϕ_2 and ϕ_3 are the associated nodal shape functions respectively.

5.6 Results

The results obtained from the present model considering all load models and displacements schemes are given in Table 1 and Table 2.

Table 1: Results obtained from the present model for the distributed traction scheme.

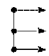
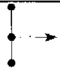
		Element size			
		**0.4 m**	**0.2 m**	**0.1 m**	**0.05 m**
U (m)$\times 10^{-5}$	1-Middle node	2.872	3.082	3.292	3.495
	2-Average 3 nodes	2.720	2.905	3.076	3.232
	3- Average continuous	2.716	2.877	3.031	3.169
V (m) $\times 10^{-5}$	1-Middle node	-1.221	-1.217	-1.216	-1.211
	2-Average 3 nodes	-1.223	-1.220	-1.219	-1.216
	3- Average continuous	-1.151	-1.142	-1.138	-1.131

Table 2: Results obtained from the present model for the virtually concentrated traction scheme.

		Element size			
		**0.4 m**	**0.2 m**	**0.1 m**	**0.05 m**
U (m) $\times 10^{-5}$	1-Middle node	2.974	3.183	3.392	3.595
	2-Average 3 nodes	2.702	2.891	3.064	3.221
	3- Average continuous	2.669	2.834	2.991	3.130
V (m) $\times 10^{-5}$	1-Middle node	-1.212	-1.215	-1.213	-1.209
	2-Average 3 nodes	-1.223	-1.220	-1.219	-1.216
	3- Average continuous	-1.136	-1.128	-1.124	-1.118

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In Table 1 and Table 2, different size of the element at the connection is also considered to allow the comparison with the different results of finite element meshes (see Table-3). In the FEM model, the used element in meshing the continuum is a 4 nodded plane elements and the skeletal element was modeled as a traditional frame.

Table 3: Results obtained from the FEM model with different mesh sizes.

	Wall Mesh				
FEM Mesh	**4×4**	**8×8**	**16×16**	**32×32**	**64×64**
U (m) $\times 10^{-5}$	2.632E-5	2.828E-5	3.162E-5	3.381E-5	3.594E-5
V (m) $\times 10^{-5}$	-1.18E-5	-1.087E-5	-1.211E-5	-1.21E-5	-1.207E-5

It can be seen that when the element size is reduced, the results approach to those of the finite element method with the more refined mesh. It has to be noted that the stiffness matrix of the present model is not symmetric but the symmetry does not affect the results. Also, the results obtained due to the displacements measurement at the middle node of the element are more close to those of the finite element method. This is mainly due to the fact that in the FEM model, the connection between the frame and the wall was simulated as single node.

6 Conclusion

In this paper, the traditional flexibility/stiffness technique was extended to couple skeletal elements with continuums modeled using the BEM. The following conclusions are presented from the implementation and the application of the present approach:

1-The the results obtained from present approach are in an excellent agreement with the results obtained from other numerical methods such as the finite element method.

2- The new technique can model the actual widths or depths of the connections between the skeletal elements and the continuums.

3-Coupling boundary element method with finite element method is done based on flexibility/stiffness analysis to gain the advantages of the assembly procedure of the well known stiffness techniques .

It can be seen that this approach can be applied to any structural problem with any number of degrees of freedom. This presented formulation can be extended to solve coupled 2D with 3D elements in which it may open the doors for newly developed software packages to allow BEM and FEM coupling. This technique could be an easy way to introduce engineers to use the BEM more extensively in their engineering applications.

7 Acknowledgement

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