A new subregion boundary element method

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Abstract

A new subregion boundary element method will be presented in this paper. In this method, the system of equations for each subregion is transformed into an appropriate form independently. The interface traction matrix equations are generated from these system of equations by using the traction and continuity conditions on the interface between subregions. The displacement components can be obtained once the unknown traction components are solved from the interface traction matrix equations. The present method is more efficient than traditional methods because it significantly reduces the size of the final matrix. Also, as the system of equations for each subregion is solved independently, parallel computing can be utilized, which is advantageous for cases where a large number of elements need to be used, such as in crack analysis. Further, if the boundary conditions are changed because the crack extension is modelled with new boundary elements, or part of the crack surfaces are in contact, only the equations related to the regions where changes of crack boundary conditions occur, are required to be recalculated. Numerical examples are presented to demonstrate the accuracy and efficiency of the method.

1 Introduction

Composite materials are used in various engineering structures. Problems such as de-bonding between the matrix and the fibres in fibre-reinforced composites, or interlaminar cracks of laminated beams often occur. The strength analysis of
composites with cracks is very important as defects or microcracks have significant influence on the load transfer behavior within the composite. Particular emphasis is placed on the evaluation of the stress intensity factor of layered materials with cracks. This stress intensity factor can be used to determine the stress and strain magnification at crack tips. Therefore, it is crucial to develop accurate and efficient techniques to calculate the stress intensity factor in layered materials with cracks.

A wide variety of analytical and numerical methods have been used to solve the fracture problems of layered materials \[1, 2, 3, 4, 5\]. If a straightforward analytical solution is not possible, numerical procedures must be resorted in order to evaluate the stress intensity factor. In general, the boundary element method (BEM) together with a subregion technique is widely considered to be a very accurate numerical tool for the analysis of problems where the materials consist of several homogeneous zones \[6, 7\]. All the boundaries of the body have to be discretised, including internal boundaries that separate homogeneous zones. The BEM equations, constructed from all homogeneous zones combined with the interface traction and continuity conditions, produce a matrix system. The numerical solution of this matrix system is the most time consuming step of the numerical method, and hence can be the bottleneck for the method being applied to problems that require large number of elements.

Kita & Kamiya \[8\] presented a special method for the subregion boundary element analysis to overcome this disadvantage. The linear system for each subregion is transformed into equations similar to the stiffness equations of the finite element method (FEM), and then the global matrix equation is constructed by superposition of these equations for each subregion. The matrix equation for each subregion is derived using the algorithm in Brebbia & Georgiou \[9\]. This algorithm can be applied easily to objects divided into subregions. The interface traction components are not obtained in the resulting matrix system, but can be calculated from the equations for the subregions. The technique has the advantage that the global coefficient matrix can be constructed easily and a smaller system of algebraic equations is obtained. This method is more effective for objects with multiple internal boundaries. However, a relatively large global coefficient matrix is still needed.

This paper presents a new numerical technique which is based on the BEM using a subregion technique. Unlike other methods which solve the displacement and traction components on the boundaries and interfaces at the same time, the distribution of traction on the interfaces is obtained first. The displacement components can then be calculated from the equations associated with the corresponding subregions. Initially, the matrix system for each subregion is transformed into a standard linear matrix form \[\mathbf{u} = \mathbf{A} \cdot \mathbf{t}\], where \(\mathbf{u}\) and \(\mathbf{t}\) are the displacement and traction vectors, respectively, and \(\mathbf{A}\) is the coefficient matrix. An interface traction matrix equation is obtained after imposing interface conditions. Extra numerical steps are
needed to set up the final interface matrix equation. However, our final matrix system is significantly smaller than the final coefficient matrix systems obtained by other methods, and thus resulting an overall numerical efficiency.

The effects of crack size, layer size, and the material properties of the composite on the stress intensity factor are then studied using the new numerical technique proposed. The dual boundary element method (DBEM) [10, 11] is incorporated into the present method to overcome the singularity in crack analysis. Further, in order to obtain improved displacement distributions around crack tips, the discontinuous quarter point element method [12, 13] is employed to calculate the stress intensity factor.

2 The new subregion boundary element method

Consider a 2-d body consisting of several subregions. For any subregion that contains no cracks, the displacement formulation of the boundary integral equation, at a boundary point \( x' \), is written in the form (the body force term is neglected)

\[
c_{ij}(x')u_i(x') + \int_{\Gamma} T_{ij}(x', x)u_j(x)d\Gamma(x) = \int_{\Gamma} U_{ij}(x', x)t_j(x)d\Gamma(x)
\]

where \( f \) stands for the Cauchy principal value integral. \( u_j(x) \) and \( t_j(x) \) are displacement and traction components in the \( j \) direction, respectively. If the boundary is smooth, \( c_{ij}(x') = \frac{1}{2} \delta_{ij} \), where \( \delta_{ij} \) is the Kronecker delta. The kernel functions \( T_{ij}(x', x) \) and \( U_{ij}(x', x) \) represent the Kelvin traction and displacement fundamental solutions, respectively, at the boundary point \( x \). For any subregion containing cracks, the DBEM is employed. The dual equations of the DBEM are the displacement and the traction boundary integral equations. The traction equation, which is applied on the crack surfaces, is obtained by differentiation of the displacement equation (1), and followed by the application of Hooke's law. It is written as

\[
\frac{1}{2} t_j(x') + n_i(x') \int_{\Gamma} S_{kij}(x', x)u_k(x)d\Gamma(x)
\]

\[
= n_i(x') \int_{\Gamma} D_{kij}(x', x)t_k(x)d\Gamma(x)
\]

where \( f \) stands for the Hadamard principal value integral, \( n_i \) denotes the \( i \)th component of the unit outward normal to the boundary, at a boundary point \( x' \). \( S_{kij}(x', x) \) and \( D_{kij}(x', x) \) are linear combinations of derivatives of \( T_{ij}(x', x) \) and \( U_{ij}(x', x) \), respectively. The displacement integral equation (1) and the traction integral equation (2) are the governing equations to be solved for the displacement on the outer boundary and the relative displacement on the crack faces.
Our new procedure is described below, which applies to a three homogeneous zone problem, as shown in Figure 1. In order to solve the integral equation numerically, the boundary is discretised into a series of elements over which displacement and traction components are written in terms of their values at a series of nodal points. Let $u_i$ and $t_i$ denote nodal displacement and traction vectors on boundary $\Gamma_i$ respectively, and $H_k$ and $G_k$ denote their coefficient matrices on the subregion $\Omega_k$. Then, for the non-cracked subregions $\Omega_1$ and $\Omega_3$, the BEM equations can be written together in matrix equations:

$$H_1 \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = G_1 \begin{bmatrix} t_1 \\ t_2 \end{bmatrix}$$

(3)

and

$$H_3 \begin{bmatrix} u_7 \\ u_8 \end{bmatrix} = G_3 \begin{bmatrix} t_7 \\ t_8 \end{bmatrix}$$

(4)

respectively. For the cracked subregion $\Omega_2$, referring to [14], the DBEM equations can be written together in a matrix equation:

$$H_2 \begin{bmatrix} u_3 \\ u_4 \\ u_5 \\ \Delta u_6 \end{bmatrix} = G_2 \begin{bmatrix} t_3 \\ t_4 \\ t_5 \\ t_6^+ \end{bmatrix}$$

(5)

where $\Delta u_6$ is the relative displacement vector on the crack surfaces and $t_6^+$ is the traction vector on the upper crack surface.

Multiplying both sides of equations (3), (4), and (5) by inverse matrices $(H_1)^{-1}$, $(H_3)^{-1}$, and $(H_2)^{-1}$, respectively, we obtain the following matrix equations

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = (H_1)^{-1} G_1 \begin{bmatrix} t_1 \\ t_2 \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix} \begin{bmatrix} t_1 \\ t_2 \end{bmatrix}$$

(6)
\[
\begin{bmatrix}
    u_7 \\
    u_8
\end{bmatrix} = (H_3)^{-1} G_3 \begin{bmatrix}
    t_7 \\
    t_8
\end{bmatrix} = \begin{bmatrix}
    c_{77} & c_{78} \\
    c_{87} & c_{88}
\end{bmatrix} \begin{bmatrix}
    t_7 \\
    t_8
\end{bmatrix} \tag{7}
\]

\[
\begin{bmatrix}
    u_3 \\
    u_4 \\
    u_5 \\
    \Delta u_6
\end{bmatrix} = (H_2)^{-1} G_2 \begin{bmatrix}
    t_3 \\
    t_4 \\
    t_5 \\
    t_6^\
\end{bmatrix} = \begin{bmatrix}
    c_{33} & c_{34} & c_{35} & c_{36} \\
    c_{43} & c_{44} & c_{45} & c_{46} \\
    c_{53} & c_{54} & c_{55} & c_{56} \\
    c_{63} & c_{64} & c_{65} & c_{66}
\end{bmatrix} \begin{bmatrix}
    t_3 \\
    t_4 \\
    t_5 \\
    t_6^\dagger
\end{bmatrix} \tag{8}
\]

where \( c_{ij} \) are the corresponding assembled matrices. Upon applying the interface traction and continuity conditions, \( t_2 = -t_3 \) and \( u_2 = u_3 \) between \( \Omega_1 \) and \( \Omega_2 \), and \( t_5 = -t_7 \) and \( u_5 = u_7 \) between \( \Omega_2 \) and \( \Omega_3 \), we obtain an interface traction matrix equation,

\[
\begin{bmatrix}
    c_{22} + c_{33} & c_{35} \\
    c_{53} & c_{55} + c_{77}
\end{bmatrix} \begin{bmatrix}
    t_3 \\
    t_5
\end{bmatrix} = \begin{bmatrix}
    c_{21} & -c_{34} & -c_{36} & 0 \\
    0 & -c_{54} & -c_{56} & c_{78}
\end{bmatrix} \begin{bmatrix}
    t_1 \\
    t_4 \\
    t_6^\dagger \\
    t_8
\end{bmatrix} \tag{9}
\]

Once the interface traction components \( t_3 \) and \( t_5 \) are solved from (9), the displacements can be calculated from the systems of equations (6), (7) and (8).

The difference between the current approach and other approaches are discussed in the next section.

### 3 Comparisons with other methods

For simplification, we consider a two-subregion problem, in which \( n_1 \) and \( n_2 \) nodes are placed on outer boundaries of the subregions, respectively, and \( m \) nodes on the interface. The traditional subregion BEM will generate an \( N \times N \) final matrix system, where \( N = 2(n_1 + n_2 + 2m) \). Kita & Kamiya's method will generate a smaller matrix system (\( \tilde{N} \times \tilde{N} \)), where \( \tilde{N} = 2(n_1 + n_2 + m) \). However, two matrices of size \( N_1 \times N_1 \) and \( N_2 \times N_2 \) need to be inverted before the final matrix can be established, where \( N_1 = n_1 + m \) and \( N_2 = n_2 + m \). In the present method, the final matrix is of size \( 2m \times 2m \) obtained from the application of the interface traction and continuity conditions. Displacement components can be calculated from the equations for the corresponding subregion. Although the inverse matrices (\( N_1 \times N_1 \) and \( N_2 \times N_2 \)) still need to be calculated first, the total number of numerical steps is less compared to both methods mentioned above for a reasonable choice of \( n_1, n_2 \) and \( m \), that would give good enough numerical accuracy.
For a problem with three or more subregions, our method is better in terms of the number of numerical steps needed regardless of the number of elements used. The savings are in the order of some multiple of $10^6$ steps. This suggests that the present method can be efficiently used to study structures of layered materials, especially those with complicated geometry.

The fact that we could calculate the matrix coefficients for each subregion independently, shows that our method is suitable for parallel computation. It also enables us to refine the boundary conditions in any subregion without having to recalculate others. This feature is very useful for numerical analysis of crack extension, and cases where crack faces come into contact.

4 Numerical results

In order to demonstrate the accuracy and efficiency of the proposed method and to show its possible applications, several examples are presented here. All the calculations are carried out under plane strain conditions with a tensile loading $T$.

In the first example, consider the stress analysis of perfectly bonded dissimilar elastic semi-strips, as illustrated in Figure 2, where $h_1 = h_2 = 2w$. 30 elements are placed on the outer boundary of each semi-stripe, and 20 elements on the interface. Two cases are considered, with $(E_2/E_1, \nu_1, \nu_2) = (9.0, 0.5, 0.5)$ and $(3.0, 0.5, 0.5)$, where $E_1$ and $E_2$ are the Young’s moduli, and $\nu_1$ and $\nu_2$ are the Poisson’s ratios, respectively. The normalised normal stress distribution on the interface are shown in Figure 3. The results agree well with published solutions.

Consider, now, a three layered plate with a crack in the centre of the middle layer as shown in Figure 4, where $h_1 = h_3 = 0.5w$, $a/w = 0.1$. The middle layer with shear modulus $\mu_2$ and Poisson’s ration $\nu_2$ is perfectly bonded between two layers having identical elastic properties $\mu_1 = \mu_3$ and $\nu_1 = \nu_3$. A crack of length $2a$ is
Figure 3: Normalised normal stress distribution on interface
\[(E_2/E_1, \nu_1, \nu_2) = (9.0, 0.5, 0.5): (a) Ref. [15], (b) the present method \]
\[(E_2/E_1, \nu_1, \nu_2) = (3.0, 0.5, 0.5): (c) Ref. [15], (d) the present method \]

located at the centre of the middle layer of thickness \( h_2 \). Discontinuous quadratic

Figure 4: Configuration of the three layered plate with a centre crack

elements are used to discretise the boundaries, 36 elements on each subregion and 6 elements on the crack surface. The stress intensity factor is normalised with respect to \( K_0 = \frac{Tr}{\sqrt{\pi a}} \), calculated for a range of ratios of \( \mu_1/\mu_2 \). \( \nu_1 = \nu_2 = \nu_3 = 0.3 \) are used in the calculations. Figure 5 shows the normalised mode I stress intensity factor versus the ratio of the shear moduli \((\mu_1/\mu_2)\) for various values of \(2a/h_2\), the ratio of crack length to layer thickness. The stress intensity factor increases as the shear modulus ratio \((\mu_1/\mu_2)\) decreases. This effect is amplified as the ratio of the crack length to layer thickness increases. It should be mentioned
that the calculated normalised stress intensity factor approaches 1 when the ratio $2a/h_2$ is small enough. This is expected, as the case is equivalent to an infinite homogeneous plate with a central crack.

Finally, consider a three layered plate with two identical co-linear cracks of length $2a$ located symmetrically in the centre of the middle layer as shown in Figure 6, where $h_1 = h_3 = 0.5w, a/w = 0.1$. The distance between the centres of the cracks is $2d = 2.4a$. The elastic properties are taken as $\mu_1 = \mu_3$ and $\nu_1 = \nu_2 = \nu_3 = 0.3$. 36 discontinuous boundary elements are placed on each subregion boundary, and 6 discontinuous elements on each of the crack. Again the

Figure 6: Configuration of the three layered plate with two identical co-linear cracks
normalised mode I stress intensity factor are calculated versus a range of the ratio of the shear modulus ($\mu_1/\mu_2$), the results are depicted in Figure 7. It is noted that the stress intensity factor increases as the shear modulus ratio ($\mu_1/\mu_2$) decreases. Further, due to the interaction between the two cracks, the mode I stress intensity factor at the crack tip $A$ is always smaller than that at the crack tip $B$.

Figure 7: Normalised mode I SIF at tips $A$ and $B$ on the three layered plate with two co-linear cracks: (a) $2a/h_2 = 0.4$, (b) $2a/h_2 = 0.6$, (c) $2a/h_2 = 0.8$, (d) $2a/h_2 = 1.0$

5 Conclusion

This paper presents a new subregion technique for boundary element analysis. In this method, the final matrix size is the number of nodal degrees of freedom over the interface. Although the inverse matrices of each subregion need to be calculated first, the total number of numerical steps in obtaining the final results are substantially less, compared to the traditional subregion BEM, and Kita & Kamiya’s subregion method, for any problem consisting of three or more sub-regions. It should be noted that for this type of problems the most time consuming and memory intensive step is solving the final matrix system. Hence, the present method has the advantages of significantly reducing computing time and memory usage compared to the above mentioned methods. Furthermore, because the system of equations for each subregion is solved independently, parallel computing can be utilized, and thus it would be advantageous to apply the method to problems with changing boundary conditions, such as, those encountered in crack analysis. Three examples were solved with this new subregion boundary element method. The results demonstrate that the present method is very effective and accurate for the boundary element analysis of an object divided into subregions.
References


