Computable representations of influence functions for multiply connected Kirchhoff plates

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Abstract

A modification is proposed of the method of functional equations which is known in solid mechanics for decades as the method of fictitious forces and, in recent days, often referred to as the method of fundamental solutions. The modification is designed to accurately compute influence functions of a point force for Kirchhoff plates containing apertures and rigid inclusions. The resolving potentials are built with the aid of the available Green's functions of the biharmonic equation for appropriate simply shaped regions. The observation and the source points of the potentials occupy different sets, resulting subsequently in functional (integral type) equations with smooth kernels. This approach allows one to obtain components of the stress tensor generated by a point force with the same accuracy level as that attained for the influence function itself.

1 Introduction

The response of a plate to a unit transverse point concentrated force is usually referred to in structural mechanics as the plate influence function. In mathematics this is associated with the Green's function of a certain boundary value problem that simulates bending of the plate. Importance of influence functions in structural mechanics is determined not only by their informative nature, but also by a wide range of their possible applications to numerical procedures like those based on the BEM.

Many of the available representations of influence functions have been obtained in the form of trigonometric series that converge but do not always allow differentiation. To illustrate the point, consider, for example, the series
expansion

$$G_1(x, y; \xi, \eta) = \frac{1}{2bD_0} \sum_{m=1}^{\infty} \frac{1 + \mu|x-\xi|}{\mu^3} e^{-\mu|x-\xi|} \sin(\mu y) \sin(\mu \eta)$$

of the influence function for Kirchhoff plate, whose midplane occupies the infinite strip-shaped region $\Omega = \{(x, y) : -\infty < x < \infty, 0 < y < b\}$, with the simply supported edges $y=0$ and $y=b$. Here $D_0$ represents the plate flexural rigidity, while the parameter $\mu$ is expressed in terms of the summation index $m$ as $\mu = m\pi/b$.

Another example of such an expansion is the classical [1] double Fourier series

$$G_2(x, y; \xi, \eta) = \frac{4}{abD_0} \sum_{k,m=1}^{\infty} \frac{\sin(\kappa x) \sin(\kappa \xi) \sin(\mu y) \sin(\mu \eta)}{(\kappa^2 + \mu^2)^2}$$

that represents the influence function for a simply supported rectangular plate ($0 \leq x \leq a$, $0 \leq y \leq b$). Here $\kappa$ and $\mu$ are expressed in terms of the summation indices $k$ and $m$ as $\kappa = k\pi/a$ and $\mu = m\pi/b$.

Although the series in eqns (1) and (2) uniformly converge (guaranteeing an accurate evaluation of the deflection function) their partial derivatives that are required for obtaining stress components caused by a point force, do not uniformly converge. Hence, the expressions of the kind in eqns (1) and (2) cannot be used to accurately compute stress components generated in the plate by a point force.

It is worth noting that only a few representations of influence functions for Kirchhoff plates that are available in the existing texts or handbooks, are compact enough and directly suitable for immediate computer implementations. Among those is the classical (see, for example, [2]) expression

$$G_3(z, \zeta) = \frac{1}{8\pi D_0} \left[ \frac{1}{2a^2} \left( a^2 - |z|^2 \right) \left( a^2 - |\zeta|^2 \right) - |z - \zeta|^2 \ln \left| \frac{a^2 - z\zeta}{a(z - \zeta)} \right| \right]$$

of the influence function for the clamped circular plate of radius $a$. Here $z$ and $\zeta$ stand for the observation and the unit force application point, respectively.

Another compact and readily computable representation of the influence function for a circular plate can be found in [3]. That is

$$G_4(z, \zeta) = \frac{1}{8\pi D_0} \left\{ |z - \zeta|^2 \ln \left| \frac{z - \zeta}{a} \right| - \frac{|a^2 - z\zeta|^2}{a^2} \ln \left| \frac{a^2 - z\zeta}{a^2} \right| \right.\\
+ \left. \frac{(a^2 - \rho^2)(a^2 - r^2)}{2a^2} \left[ \frac{1 + \omega}{\omega} - 2\omega \sum_{m=1}^{\infty} \frac{1}{m(m+\omega)} \frac{r^\rho^m}{a^2} \cos(m(\phi - \psi)) \right] \right\}$$

the influence function for a simply supported plate of radius $a$. The parameter $\omega$ is defined here in terms of the Poisson ratio of the material as $\omega = (1+\sigma)/2$. 
2 Multiply connected plates

An efficient procedure is developed in this study for obtaining the response of a multiply connected Kirchhoff plate to a transverse point force. The procedure is based on a modification of the method of functional equations [4]. In recent publications, this method is often referred to as the method of fundamental solutions [5].

The suggested in [6] modification of the method of functional equations uses some existing representations of influence functions for simply connected plates of standard shape. Only differentiable representations of standard influence functions are acceptable. That is, representations of the kind in eqns (3) and (4) are applicable to this study, in contrast to those in eqns (1) and (2), which can potentially be used for computer applications, but only after some preliminary analytical adjustments.

To introduce the basic idea of the algorithm for computing influence functions for multiply connected Kirchhoff plates, we consider an either closed or semi-closed doubly connected region $D$ whose outer boundary $C$ is piecewise smooth, while the aperture's edge $L$ represents a simple smooth closed curve.

We consider the following homogeneous boundary value problem

$$\Delta \Delta w(x, y) = 0, \quad (x, y) \in D$$

$$B_{1C}[w(x, y)] = B_{2C}[w(x, y)] = 0, \quad (x, y) \in C$$

$$B_{1L}[w(x, y)] = B_{2L}[w(x, y)] = 0, \quad (x, y) \in L$$

Here $\Delta$ is the Laplace operator. The discussion is limited to the most practical cases, where each segment of the boundary lines $C$ and $L$ is exposed to either free edge, or simple/elastic clamp, or simple/elastic support condition.

In addition, we assume that the problem in eqns (5)-(7) has only the trivial solution or, in other words, the problem in eqns (6) and (7) stated for a nonhomogeneous biharmonic equation has a unique solution. This implies that the plate's state, simulated by the problem, is physically feasible, which, in turn, assures the existence and uniqueness of its Green's function.

Let $G_0(x, y; \xi, \eta)$ be the Green's function to the boundary value problem posed with eqns (5) and (6), stated over the simply connected region $D^*$ bounded with $C$. And let the Green's function $G(x, y; \xi, \eta)$ to the problem posed with eqns (5)-(7) be presented over the doubly connected region $D$ as

$$G(x, y; \xi_0, \eta_0) = G_0(x, y; \xi_0, \eta_0) + g(x, y)$$

where $(\xi_0, \eta_0)$ is an arbitrary point in $D$. From this representation, it follows that the component $g(x, y)$ ought to be a biharmonic function everywhere in $D$ and, in addition, it ought to satisfy the following boundary conditions

$$B_{1C}[g(x, y)] = B_{2C}[g(x, y)] = 0, \quad (x, y) \in C$$
In what follows, \( g(x, y) \) is referred to as the regular component of the influence function. Being motivated by the key concept of the method of functional equations [4], we look for \( g(x, y) \) in the following modified potential form

\[
g(x, y) = \int_{F} G_{0}(x, y; \xi, \eta) q_{1}(\xi, \eta) \, dF(\xi, \eta) + \int_{F} \frac{\partial G_{0}(x, y; \xi, \eta)}{\partial \nu} q_{2}(\xi, \eta) \, dF(\xi, \eta), \quad (x, y) \in D
\]

where \( F \) is a closed curve contained in \( L \), \( \nu \) is the normal direction to \( F \) at \((\xi, \eta)\), and the differentiation with respect to \( \nu \) is understood as

\[
\frac{\partial}{\partial \nu} = \cos(\nu, \xi) \frac{\partial}{\partial \xi} + \cos(\nu, \eta) \frac{\partial}{\partial \eta}
\]

The density functions \( q_{1}(\xi, \eta) \) and \( q_{2}(\xi, \eta) \) in eqn (12) are assumed integrable on \( F \). In this study, the latter is referred to as the fictitious contour.

Note that, since the source point set \((\xi, \eta)\) of the potential representation in eqn (12) lies outside \( D \), the kernels \( G_{0}(x, y; \xi, \eta) \) and \( \frac{\partial}{\partial \nu} G_{0}(x, y; \xi, \eta) \), as functions of the observation point \((x, y)\): (i) are biharmonic in \( D \) and (ii) satisfy the boundary conditions of eqn (9). These properties hold for any integrable density functions \( q_{1}(\xi, \eta) \) and \( q_{2}(\xi, \eta) \). This justifies the fact that \( g(x, y) \), as defined by eqn (12), is also biharmonic in \( D \) and satisfies the boundary conditions in eqn (9).

The density functions \( q_{1}(\xi, \eta) \) and \( q_{2}(\xi, \eta) \) can be determined by satisfying the boundary conditions imposed by eqns (10) and (11). In going through that procedure, one must keep in mind that, since the kernel functions \( G_{0}(x, y; \xi, \eta) \) and \( \frac{\partial}{\partial \nu} G_{0}(x, y; \xi, \eta) \) are biharmonic in \( D \), all their partial derivatives implied by the boundary operators \( B_{1L} \) and \( B_{2L} \) are continuous on \( L \). Hence, when substituting the expression from eqn (12) in the relation of eqn (10), one obtains

\[
-B_{1L}[G_{0}(x, y; \xi_{0}, \eta_{0})] = \int_{F} B_{1L}[G_{0}(x, y; \xi, \eta)] q_{1}(\xi, \eta) \, dF(\xi, \eta)
\]

\[
+ \int_{F} \frac{\partial}{\partial \nu} \left( B_{1L}[G_{0}(x, y; \xi, \eta)] \right) q_{2}(\xi, \eta) \, dF(\xi, \eta), \quad (x, y) \in L, \quad (13)
\]
while the following relation results from eqn (11)

\[-B_{2L}[G_0(x, y; \xi_0, \eta_0)] = \int_F B_{2L}[G_0(x, y; \xi, \eta)] q_1(\xi, \eta) \, dF(\xi, \eta)\]

\[+ \int_F \frac{\partial}{\partial \eta} \left( B_{2L}[G_0(x, y; \xi, \eta)] \right) q_2(\xi, \eta) \, dF(\xi, \eta), \quad (x, y) \in L \quad (14)\]

The relations in eqns (13) and (14) represent a system of functional (integral type) equations in the density functions \(q_1(\xi, \eta)\) and \(q_2(\xi, \eta)\).

Since the set \((\xi, \eta)\) of integration points in eqn (12) is moved out of \(D\), the functions \(G_0(x, y; \xi, \eta)\) and \(\frac{\partial}{\partial \eta} G_0(x, y; \xi, \eta)\) are regular in \(D\) and cause no singularity for the kernel functions of the functional equations (13) and (14). This is a "good and bad news" at the same time (good - because regular integrals are easier to evaluate numerically, compared to singular ones, bad - because numerical procedures of solving regular integral equations of the first kind are not usually stable).

Thus, with a readily computable representation of \(G_0(x, y; \xi_0, \eta_0)\) in place, the numerical solution of the system in (13) and (14) should not be a problem. Indeed, all the required derivatives of \(G_0(x, y; \xi_0, \eta_0)\) can be computed analytically and the only practical concern that remains is how to specify and locate \(F\). No general recommendations can be suggested for that. Note, nevertheless, that, as our experience advices, if a certain class of shapes is selected, then the procedure for specifying \(F\) could be formalized.

### 3 Validation example

The decisive computational parameters of the procedure sketched in Section 2 are associated with the approximate evaluation of the line integrals in eqns (12)-(14) and with the choice of the fictitious contour \(F\). To show how optimal values of these parameters can be found, a validation problem is considered having an exact solution. We shall determine a function \(w(x, y)\) that is biharmonic in the semi-strip \(D = (0 < x < \infty, 0 \leq y \leq b)\) having a smooth opening \(L\), with the boundary conditions imposed as

\[w(x, 0) = w(x, b) = 0; \quad \frac{\partial^2 w(x, 0)}{\partial y^2} = \frac{\partial^2 w(x, b)}{\partial y^2} = 0\]

\[\frac{\partial^2 w(0, y)}{\partial x^2} + \sigma \frac{\partial^2 w(0, y)}{\partial y^2} = 0\]

\[\frac{\partial}{\partial x} \left( \frac{\partial^2 w(0, y)}{\partial x^2} + (2-\sigma) \frac{\partial^2 w(0, y)}{\partial y^2} \right) = 0\]
Let \( G_S(x, y; E, q) \) be the influence function for the plate occupying the region \( D \), with the edges \( y=0 \) and \( y=b \) simply supported, and the edge \( x=0 \) free of tension. The following representation

\[
G_S(x, y; \xi, \eta) = \frac{1}{2bD_0} \sum_{m=1}^{\infty} \left[ \frac{1}{\mu^3} \left( (1+\mu|x-\xi|) e^{-\mu|x-\xi|} \right) \right. \\
+ \left. \left( \frac{4+(1+\sigma)^2 + \mu(1-\sigma)^2(x+\xi + 2\mu x\xi)}{(1-\sigma)(3+\sigma)} e^{-\mu(x+\xi)} \right) \right] \sin(\mu y)\sin(\mu \eta)
\]  

(20)

for \( G_S(x, y; \xi, \eta) \) can be found in [3].

Clearly, if the functions \( \theta_1(x, y) \) and \( \theta_2(x, y) \) in eqn (19) represent the traces of \( G_S(x, y; \xi^*, \eta^*) \) and its normal derivative on \( L \), then \( G_S(x, y; \xi^*, \eta^*) \), as a function of \( x \) and \( y \), is the exact solution to the problem posed by eqns (15)-(19) in \( D \), given the source point \((\xi^*, \eta^*)\) does not belong to \( D \).

The data in Table 1 reveal the accuracy level attained for the problem posed with eqns (15)-(19). In this setting, the geometrical and physical parameters were chosen as follows: the semi-strip's width \( b=\pi \), the aperture \( L \) is a circle of radius \( R=1.0 \) centered at \((2.5, \pi/2)\), the source point \((\xi^*, \eta^*)\) is fixed inside \( L \) at \((3.0, \pi/2)\), and the Poisson ratio of the material \( \sigma=0.3 \).

<table>
<thead>
<tr>
<th>Test point</th>
<th>( w )</th>
<th>( \partial^2 w/\partial x^2 )</th>
<th>( \partial^2 w/\partial y^2 )</th>
</tr>
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<tbody>
<tr>
<td>((0, \pi/2))</td>
<td>0.225065</td>
<td>0.067620</td>
<td>-0.225402</td>
</tr>
<tr>
<td>((0.3, \pi/2))</td>
<td>0.241616</td>
<td>0.086702</td>
<td>-0.242215</td>
</tr>
<tr>
<td>((0.6, \pi/2))</td>
<td>0.265923</td>
<td>0.099617</td>
<td>-0.267133</td>
</tr>
<tr>
<td>((0.9, \pi/2))</td>
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<td>0.106761</td>
<td>-0.301745</td>
</tr>
<tr>
<td>((1.2, \pi/2))</td>
<td>0.341940</td>
<td>0.106791</td>
<td>-0.347561</td>
</tr>
<tr>
<td>((1.5, \pi/2))</td>
<td>0.394261</td>
<td>0.096447</td>
<td>-0.406396</td>
</tr>
</tbody>
</table>

Note that in obtaining the data for Table 1, the highest accuracy level was attained if the radius \( R_0 \) of the fictitious contour \( F \) equals 0.6. The system of functional equations (13) and (14) has been numerically solved by the standard trapezoidal rule, with 20 quadrature nodes uniformly spaced on \( F \). The series expansions in the influence function \( G_S(x, y; \xi, \eta) \) and of all of its derivatives have been evaluated by their twenty fifth Fourier polynomials.
4 Some applications

If a readily computable representation of the influence function is available, then computing of the components of the plate's stress-strain state, caused by a transverse point force, represents the direct use of the influence function. On the other hand, there exist many other possible applications of these functions. Some of those are tackled in this section illustrating the effectiveness of the approach.

It can readily be shown how the approach might be extended to a plate that is subject, for example, to a set of point concentrated forces of different magnitudes applied at different locations. A word of caution is appropriate. The point is that such an extension is feasible if the resultant plate's state remains linear geometrically as well as physically.

Consider a plate occupying the infinite strip-shaped region \( D \) of width \( b \), with simply supported edges \( y = 0 \) and \( y = b \), weakened with an aperture, whose edge \( L \) is free of tension. Let the plate be subject to a finite set of \( k \) transverse point concentrated forces of magnitudes \( p_i \) applied to locations \( (\xi_i, \eta_i) \), \( (i = 1, \ldots, k) \).

By the superposition principle, the reaction of the plate to the set of forces \( p_i \) can be found as

\[
w(x, y) = \sum_{i=1}^{k} p_i G_1(x, y; \xi_i, \eta_i) + g(x, y),
\]

where \( G_1(x, y; \xi_i, \eta_i) \) is the influence function shown in eqn (1). The regular component \( g(x, y) \) in the above equation can then be found by the procedure described in Section 2.

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Figure 1: Plate with an aperture subject to two point forces
The profile of the deflection function \( w(x, y) \) is shown in Figure 1 for the plate of width \( \pi \) having a symmetrically spaced circular hole of radius 0.8 and subject to two opposite unit forces applied to points \((2.0, \pi/2)\) and \((3.5, \pi/2)\). The deflection is depicted along the symmetry line \( y = \pi/2 \), which passes through the force application points. Note that the optimal fictitious contour \( F \) was found in this case as the circle of radius 0.75 concentric with the aperture.

Figure 2: Deflection of the strip-shaped plate with a fixed inclusion

Figure 2 depicts the deflection function caused by two point forces of magnitude \( p_1 = 2 \) and \( p_2 = 1 \) applied at \((0.2, 0.75)\) and \((-0.6, 0.2)\), respectively, in the infinite strip-shaped plate of width \( b = 1 \), having a fixed rigid circular inclusion of radius \( a = 0.25 \) centered at \((0, 0.3)\). The optimal fictitious contour \( F \) was found in this case as a circle of radius \( R_0 = 0.24 \) concentric with \( L \).

For our next example, we consider a plate, whose midplane occupies the semi-infinite strip with an absolutely rigid circular inclusion \( L \) of radius \( R \) centered at \((x_0, y_0)\). The edges \( y = 0 \) and \( y = \pi \) are simply supported, while the edge \( x = 0 \) is free of tension. The deflection is sought as a biharmonic function \( w(x, y) \) satisfying the boundary conditions imposed by eqns (15)-(19), where \( b = \pi \).

We look for \( w(x, y) \) in the following form

\[
w(x, y) = \int_F G_5(x, y; \xi, \eta) \, q_1(\xi, \eta) \, dF(\xi, \eta) \\
+ \int_F \frac{\partial G_5(x, y; \xi, \eta)}{\partial \nu} \, q_2(\xi, \eta) \, dF(\xi, \eta), \quad (x, y) \in D,
\]
The density functions $q_1(\xi, \eta)$ and $q_2(\xi, \eta)$ of this representation are to be found upon satisfying the boundary conditions imposed on the inclusion contour in compliance with eqn (19).

Let $t$ be the angular coordinate of a local polar coordinate system associated with the center $(x_0, y_0)$ of the inclusion $L$. Then the case of $\theta_1 = a_0 \cos(t)$ and $\theta_2 = (a_0/R) \cos(t)$ in eqn (19) can be interpreted as the setting, in which the inclusion is turned the angle $\arctan(a_0/R)$ about the axis $x = x_0$ off the plate's midplane. Some components of the plate stress-strain state, computed for this setting on the line $y = y_0$, are shown in Figure 3 for the following set of parameters: $x_0 = 3.0$, $y_0 = \pi/2$, $R = 1.0$, $a_0 = 0.1$.

![Figure 3: Circular inclusion turned off the plate's middle plane](image)

As we have pointed out earlier in Section 2, the procedure of finding the optimal location and configuration of the fictitious contour $F$ can hardly be formalized in general. However, as this study reveals, if the shape of an actual aperture/inclusion $L$ is a circle, then the range of possible shapes of $F$ can also be limited to circles concentric with $L$. The radius of $F$ can easily be then determined on the case by case basis.

5 Closure

To make plate influence functions suitable for engineering implementations, one has to address two key concerns. First, what accuracy level is attainable in computing influence functions and their derivatives. Addressing this issue, an effective algorithm was developed in this study for obtaining computable representations of influence functions. The second practical concern
is, whether such representations of influence functions are applicable in solving some relevant problem classes from plate theory. The present study illustrates the practicality of the influence function based method in solving a selective type of plate problems.

References