Analysis of general time-dependent problems with the hybrid boundary element method

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Abstract

More than three decades ago, Przemieniecki [1] introduced a formulation for the free vibration analysis of bar and beam elements based on a power series of frequencies. Recently, this formulation was generalized for the analysis of the dynamic response of elastic systems submitted to arbitrary nodal loads as well as initial displacements [2]. Based on the mode-superposition method, a set of coupled, higher-order differential equations of motion is transformed into a set of uncoupled second order differential equations, which may be integrated by means of standard procedures. Motivation for this theoretical achievement is the hybrid boundary element method [3, 4], as developed in [2] for time-dependent problems on the basis of a frequency-domain formulation, which, as a generalization of Pian’s previous achievements for finite elements [5], yields a stiffness matrix that requires only boundary integrals, for arbitrary domain shapes and any number of degrees of freedom. The use of higher-order frequency terms drastically improves numerical accuracy. The introduced modal assessment of the dynamic problem is applicable to any kind of finite element for which a generalized stiffness matrix is available [6, 7]. The present paper is an attempt of consolidating this boundary-only theoretical formulation, in which a series of particular cases are conceptually outlined and numerically assessed: Constrained and unconstrained structures; initial displacements and velocities as nodal values as well as prescribed domain fields (including rigid body movement); forced time-dependent displacements; self-weight and domain forces other than inertial forces; evaluation of results at internal points. Two academic examples for 2D problems of potential illustrate the formulation.
1 Problem formulation in the frequency domain

The time effect to be considered in the theoretical outline is due to the inertia of an elastic body [2]. One is attempting to find the displacement field $u_i$ with a corresponding stress field $\sigma_{ij}$ that satisfies the equilibrium partial differential equation, for a given circular frequency $\omega$ [4, 7],

$$\sigma_{ij} + \dot{f}_i + \omega^2 \rho u_i = 0$$

in the domain $\Omega$, for applied body forces $\ddot{f}_i$ and specific body mass density $\rho$. Subscripts $i$ and $j$ may assume values 1, 2, and 3, as referred to global coordinates $x$, $y$, and $z$, respectively. A subscript after a comma denotes derivative with respect to the corresponding coordinate direction. Repeated indices indicate a three-term-summation, in the general case of three-dimension problems.

The displacements must satisfy the boundary condition for prescribed boundary displacements $\ddot{u}_i$. Also, the stress field must be in equilibrium with prescribed forces $\ddot{t}_i$ along the complementary part $\Gamma_\sigma$ of the boundary:

$$u_i = \ddot{u}_i \text{ along } \Gamma_u \text{ and } \sigma_{ij} \eta_j = \ddot{t}_i \text{ along } \Gamma_\sigma.$$  

A solution exactly satisfying all three equations above is possible only in certain particular cases. One assumes that a displacement field

$$\ddot{u}_i = u_{in} d_m(\omega) \text{ along } \Gamma$$

such that $\ddot{u}_i = \ddot{u}_i$ along $\Gamma_u$ is known along the boundary in terms of polynomial interpolation functions $u_{im}$ and some frequency-dependent nodal displacement parameters $d_m(\omega)$, in which $m$ refers to each one of the degrees of freedom of the discretized model.

One also assumes a different displacement field

$$u_i = u^*_i + u^{p*}_i \text{ in } \Omega$$

for the entire domain, in such a way that the equilibrium eqn (1) is identically satisfied. It means that one can define an arbitrary particular solution $u^*_i$, such that the corresponding stress field $\sigma^*_i$ satisfies the equation

$$\sigma^*_{ij} + \ddot{f}_i + \omega^2 \rho u^*_i = 0 \text{ in } \Omega$$

and, most important, it means that one can find a homogeneous solution $u^*_i$ with corresponding stress field $\sigma^*_{ij}$ that satisfies identically

$$\sigma^*_{ij} + \omega^2 \rho u^*_i = 0 \text{ in } \Omega.$$  

This characterizes a fundamental solution

$$u^*_i = u^*_{i_m}(\omega) p^*_m(\omega) \quad \text{and} \quad \sigma^*_i = \sigma^*_{i_m}(\omega) p^*_m(\omega),$$

with $m$ refering to each one of the degrees of freedom of the discretized model.
Given these assumptions, one shall look for a means of relating the fields \( \tilde{u}_i \), defined by eqn (3), and \( u_i \), defined by eqns (4) – (7), in such a way that eqn (1) is best satisfied. This is achieved by means of a variational principle [2], in terms of the Hellinger-Reissner potential, generalized for frequency-dependent problems,

\[
- \int \left( \delta \sigma_{\gamma} + \rho \omega^{2} \delta u_i, (u_i - \tilde{u}_i) \right) \partial \Omega + \int \delta \sigma_{\gamma} (u_i - \tilde{u}_i) d\Omega + \int \delta \tilde{u}_i \left( \sigma_{\gamma} + \rho \omega^{2} u_i \right) d\Omega - \sum \delta \tilde{u}_i \left( \sigma_{\gamma} - i \right) d\Gamma = 0
\]  

(8)

After interpolation of the variables \( \tilde{u}_i \), according to eqn (3), as well as \( u_i \) and \( \sigma_{\gamma} \), according to eq (7), one arrives at the expression

\[
\delta p^T (Fp^* - Hd + b) - \delta d^T (H^T p^* + p^b - p) = 0
\]  

(9)

This equation is expressed in matrix notation, for convenience. The quantities \( p^* \) and \( d \) are vectors containing the nodal parameters \( p_m^* \) and \( d_m \), respectively – the primary unknowns of the problem. The symmetric flexibility matrix \( F \), the cinematic transformation matrix \( H \) and the vector \( b \) of nodal displacements equivalent to body forces are defined in terms of boundary integrals as

\[
\begin{bmatrix}
F \\
H^T \\
B^T
\end{bmatrix}
= 
\begin{bmatrix}
F_{nn} \\
H_{mn} \\
b_m
\end{bmatrix} = 
\begin{bmatrix}
u_m^* \\
u_n^* \sigma_m^* \\
u_n^* \delta_m
\end{bmatrix} d\Gamma + 
\begin{bmatrix}
u_m^* \\
u_n^* \sigma_m^* \\
u_n^* \delta_m
\end{bmatrix} d\Gamma
\]  

(10)

Owing to the singularity of the fundamental solution, the boundary integral represented above is singular and has to be split into a Cauchy principal value and a discontinuous term. Related to this singularity, a generalized Kronecker delta is introduced, meaning that \( \delta_{im} = 0 \) in general, except if the indices \( i \) and \( m \) refer to the same degree of freedom, when \( \delta_{im} = 1 \).

In eqn (9), \( t \) and \( p \) are vectors of nodal forces equivalent to body forces \( \tilde{f}_i \) and traction forces \( \tilde{t}_i \), respectively, defined as

\[
[p^b \ p] = [p_m^b \ p_m] = \int \left[ u_m^b \sigma_m^* \tilde{t}_i \right] d\Gamma
\]  

(11)

Moreover, virtual work considerations [7] enable writing \( b = Hd^b \) in which \( d^b \) are displacements \( u_i^b \) measured directly at the boundary points. Then, for arbitrary values of \( \delta p^* \) and \( \delta d \), eqn (9) becomes

\[
Fp^* - H(d - d^b) = 0 \quad \text{and} \quad H^TP^* - (p - p^b) = 0
\]  

(12)

or

\[
H^TF^{-1}H(d - d^b) = p - p^b
\]  

(13)

in which

\[
H^TF^{-1}H = K
\]  

(14)

constitutes a symmetric, positive semi-definite stiffness matrix that transforms nodal displacements \( d \) into nodal forces in equilibrium with the set of equivalent
Equations (9) – (14), as established above for frequency-dependent problems, are formally the same ones obtained by the first author for static problems [3, 4], which, in absence of body forces, also formally coincides with the hybrid finite element formulation [5].

The fundamental solution (7) may be adequately expressed as

\[ u_{m}^{*}(\omega) \leftarrow u_{m}^{*}(0) + u_{m}^{*}(\omega) \quad \text{and} \quad \sigma_{qn}^{*}(\omega) \leftarrow \sigma_{qn}^{*}(0) + \sigma_{qn}^{*}(\omega) \]  

(15)

in which \( u_{m}^{*}(0) \) and \( \sigma_{qn}^{*}(0) \) correspond to the static fundamental solution. The frequency-dependent terms \( u_{m}^{*}(\omega) \) and \( \sigma_{qn}^{*}(\omega) \) are by construction non-singular, real functions that require no special consideration for the sake of integration, as expressed in eqn (10). For two-dimensional problems of potential, for example, the Helmholtz equation corresponding to eqn (6) has as solution [7]

\[ \theta^{*} = -\frac{1}{2\pi} \ln(r) + \frac{1}{2\pi} \left( \frac{\pi}{2} BesselY(0, kr) - \ln(r) - \left( \ln \left( \frac{k}{2} \right) + \gamma \right) BesselJ(0, kr) \right) \]  

(16)

As a consequence of writing the fundamental solution as in eqn (15), the matrices \( F \) and \( H \) of eqn (10) may be formally represented as

\[ F(\omega) = F_{0} + F_{\omega}, \quad H(\omega) = H_{0} + H_{\omega} \]  

(17)

The frequency-dependent terms \( F_{\omega}, H_{\omega} \) involve no singularities. On the other hand, the terms \( F_{0}, H_{0} \) correspond to the matrices of a static formulation, with integration singularities that must be dealt with adequately. Moreover, the terms about the main diagonal of \( F_{0} \), which cannot be evaluated by means of the integration indicated in eqn (10), are obtained from static properties considerations, for both cases of finite and infinite regions [2, 4, 7].

2 General time-dependent analysis in the frequency domain

Instead of formulating a problem for a given frequency, one may express the fundamental solution, eqn (15), as a power series of frequencies. For example, eqn (16) would read

\[ \theta^{*} = \frac{\ln(r)}{2\pi} + \frac{k^{2}r^{2}}{27648\pi} \left[ \left( \ln(r) - 1 \right) 3456 - (216 \ln(r) - 324) k^{2}r^{2} + (6 \ln(r) - 11) k^{4}r^{4} \right] + O(r^{8}) \]  

(18)

As a consequence, the matrices \( F \) and \( H \), defined in eqn (10), as well as \( K \), defined in eqn (14), also become power series of frequencies with an arbitrary number \( n \) of terms [4, 7]:

\[ F = \sum_{i=0}^{n} \omega^{i} F_{i}, \quad H = \sum_{i=0}^{n} \omega^{i} H_{i}, \quad K = \sum_{i=0}^{n} \omega^{i} K_{i} \]  

(19)

Introducing a time function \( \tau(t, \omega) \) such that \( \partial^{2} \tau(t, \omega)/\partial^{2}t = -\omega^{2} \tau(t, \omega) \), one may compose a time-dependent vector \( D \) of nodal displacements as a truncated
series with \( m \) terms (the specific aspect of this series is not relevant):

\[
d = d(t) = \sum_{j=1}^{\infty} d_j \tau(t, \omega_j)
\]  

(20)

Then, it is possible to model the behavior of a damping-free structure as

\[
\sum_{j=1}^{\infty} \left( K_0 - \sum_{i=1}^{\infty} \omega_j^2 M_i \right) \left( d_j - d^+_j \right) \tau(\omega_j) = p(t) - p^+(t)
\]  

(21)

In this equation, one expresses \( K_0 \) explicitly as the stiffness matrix of the static discrete-element formulation and renames the remaining terms of the power series of \( K \) in eqn (19) as \(-M_j\), as generalized mass matrices, although they constitute a blending of mass and stiffness matrices. The only exception is the matrix \( M_1 \), which corresponds to the mass matrix obtained in the conventional formulation that truncates after \( \omega^2 \) [1]. The vectors \( d_j \) of displacements are the unknowns of the problem, to be determined for applied body and traction forces as well as initial nodal displacements and velocities. The number \( n \) of frequency-related matrices is arbitrary. The advantage of such a formulation based on a power series of frequencies is that it provides a more accurate fulfillment of the dynamic differential equilibrium equations of stresses at internal points of the elastic body [2, 7].

According to the definition of \( \tau(t, \omega) \), eqn (21) may be rewritten as

\[
\left( K_0 - \sum_{i=1}^{n} (-1)^i M_i \frac{\partial \omega^i}{\partial t^2} \right) (d - d^+) = p(t) - p^+(t)
\]  

(22)

which is a coupled set of higher-order time derivatives that makes use of the matrices obtained in the frequency formulation.

Let the non-linear eigenvalue problem related to equation above

\[
K_0 \Phi - \sum_{i=1}^{n} M_i \Phi \Omega^2 = 0
\]  

(23)

be solved, in which \( \Omega^2 \) is a diagonal matrix with as many eigenvalues \( \omega^2 \) as required to model the structural behavior and \( \Phi \) is a matrix whose columns are the corresponding eigenvectors, satisfying the orthogonality conditions [2, 7]:

\[
\sum_{i=1}^{n} \sum_{j=1}^{n} \Omega^{2j-2i} \Phi^T M_j \Phi \Omega^{2i-2} = I,
\Phi^T K_0 \Phi + \sum_{i=1}^{n} \sum_{j=1}^{n} \Omega^{2j-2i} \Phi^T M_j \Phi \Omega^{2i-2} \right) = \Omega^2
\]  

(24)

In the context of a mode-superposition procedure, one may approximate the time-dependent displacements \( d(t) \) as a finite sum of contributions of the normalized eigenvectors \( \Phi \) (according to equations above) multiplied by a vector of amplitudes \( \eta = \eta(t) \), which are the new unknowns of the problem:

\[
d = \Phi \eta
\]  

(25)

After some tedious manipulation [2, 7], one transforms eqn (22) into

\[
\Omega^2 (\eta - \eta^+) + \dot{\eta} - \ddot{\eta}^+ = \Phi^T (p - p^+)
\]  

(26)
an uncoupled system of second order differential equations of time, which may be integrated by means of standard procedures.

For the sake of both expressing the initial displacement conditions necessary as integration constants in equations above and obtaining \( \eta^b \) from \( d^b \), it may be demonstrated \([7]\) that, for the subsets \( \Phi_{el} \) and \( \Omega_{el} \) of modes and frequencies related to plain elastic deformation,

\[
\eta_{el} = [\Phi_{el}^T K_{el} \Phi_{el}]^{-1} \Phi_{el}^T K_{el} d
\]

(27)

whereas for the subsets \( \Phi_{rig} \) and \( \Omega_{rig} = 0 \) related to rigid body displacements,

\[
\eta_{rig} = \Phi_{rig}^T M_{rig} d
\]

(28)

The set of uncoupled second order differential equations of time (26), together with eqns (27) and (28) for the consideration of initial displacements, is the transformation of eqn (22) for the solution of a wide range of time-dependent problems on the basis of a frequency formulation.

2.1 Consideration of forced nodal displacements

When part of the nodal displacements are known functions of time, one proceeds exactly as in the conventional dynamic analysis \([1, 2, 7]\), rewriting eqn (22) in terms of submatrices:

\[
\begin{pmatrix}
K_{pp} & K_{pf} \\
K_{fp} & K_{ff}
\end{pmatrix}
- \sum_{i=1}^{n} (-1)^{i} \begin{pmatrix}
M_{pp} & M_{pf} \\
M_{fp} & M_{ff}
\end{pmatrix} \frac{\partial^{2i}}{\partial t^{2i}} \begin{pmatrix}
d - d^b \\
d - d^b
\end{pmatrix}^{\prime} = \begin{pmatrix}
(p - p^b) \prime \\
(p - p^b) \prime
\end{pmatrix}
\]

(29)

in which the subscripts \( p \) and \( f \) refer to prescribed and free subsets of nodal displacements, respectively. The second line of submatrix equations above reads

\[
\begin{pmatrix}
K_{pp} & K_{pf} \\
K_{fp} & K_{ff}
\end{pmatrix}
- \sum_{i=1}^{n} (-1)^{i} M_{pp} \frac{\partial^{2i}}{\partial t^{2i}} \begin{pmatrix}
d - d^b \\
(d - d^b) \prime
\end{pmatrix} = \begin{pmatrix}
(p - p^b) \prime \\
(p - p^b) \prime
\end{pmatrix}
\]

(30)

Since all quantities at the right-hand side are known functions of time, this equation is formally equivalent to eqn (22), for the sake of the modal superposition procedure used to arrive at eqn (26). Once the displacements \( d' \) are obtained, after transformation of eqn (30) in a set of uncoupled second order equations, the time-dependent reaction forces related to the prescribed nodes may be evaluated using the first line of the submatrix equations (29). A word of caution is needed concerning the numerical implementation of eqn (30), as the higher-order derivatives present at the right-hand side may lead to weird results if \( (d - d^b) \prime \) is a time series approximation \([4]\).

2.2 Evaluation of results at internal points

In a harmonic formulation, the singular force parameters \( p^\prime(\omega) \) are obtained, according to the first of eqns (12), by

\[
p^\prime(\omega) = S(\omega)(d(\omega) - d^b(\omega))
\]

(31)
in which \( S(\omega) = F^{-1}(\omega)H(\omega) \). Transformed to the time-dependent formulation introduced in this section, and considering forced nodal displacements, for the sake of generality, eqn (31) reads [6, 7]

\[
p'(t) = \sum_{i=0}^{n} S_i' \Phi \Omega^{2i} (\eta - \eta^0) + \sum_{i=0}^{n} (-1)^i S_i^e \frac{\partial^{2i}}{\partial t^{2i}} (d - d^0)
\]

(32)
in which the matrices \( S_i \) have been split in submatrices related to free and prescribed nodal displacements. It is worth observing that the only higher-order derivatives involved are related to the prescribed nodal displacements, which must be evaluated as accurately as possible.

Once \( p'(t) \) is evaluated, one is able to represent results at internal points. For this sake, consider the general displacement expression in the first of eqns (7) and the corresponding superposition of effects in the frequency domain [4, 7]:

\[
u(t) = \sum_{j=0}^{n} \sum_{i=0}^{\nu} \omega^{2i} u_j' p^j = \sum_{i=0}^{n} u_j' \Omega^{2i} p^j
\]

(33)

Note that \( u(t) \) is a vector with as many elements as the number of degrees of freedom at a domain point (one for problems of potential, two for 2D elasticity and 3 for 3D elasticity). Also, each matrix \( u_j' \) has as many rows as a point's degrees of freedom and as many columns as the problem's nodal degrees of freedom. Making use of this equation, the expression of \( u \) becomes, for the general consideration of body forces and forced nodal displacements [7]:

\[
u(t) = \sum_{j=0}^{n} \sum_{i=0}^{\nu} u_j' S_j' \Phi \Omega^{2i} (\eta - \eta^0) + \sum_{i=0}^{n} (-1)^i \sum_{j=0}^{\nu} u_j' S_j^e \frac{\partial^{2i}}{\partial t^{2i}} (d - d^0)
\]

(34)

In this equation, only exponentials of \( \Omega \) and derivatives of \( d \) up to \( 2n \) are retained in the expressions. As a matter of illustration, eqn (34) reads, for \( n = 3 \), in absence of body forces and considering no prescribed nodal displacements,

\[
u(t) = \left[u_0 S_0 \Phi + (u_0' S_1 + v_0 S_0) \Phi \Omega^2 + (u_2 S_2 + v_1 S_1 + u_0 S_0) \Phi \Omega^4 + (u_3 S_3 + v_2 S_2 + u_1 S_1 + v_0 S_0) \Phi \Omega^6 \right] \eta
\]

(35)

Equation (34) is the generalization of a traditional procedure in the dynamic analysis, for prescribed nodal displacements [1].

3 Two validating patch tests

3.1 Cutout of a drumhead submitted to initial velocity

Consider the evaluation of the displacements of a drumhead, as shown in the upper left of Fig. 1, submitted to initial velocity \( v_0 = (r-1)\sin(\theta) \). This problem can be solved in the frame of the theory of potential [8], as illustrated in the snapshot in the lower right of the figure. As a simple academic exercise, a patch is cut out and discretized with a total of 34 equally spaced quadratic elements. Zero displacements are prescribed along the drumhead’s curved edge. Time-dependent,
equivalent nodal forces, as evaluated for the solution of the whole circular region, are applied at the remaining boundary nodes. The displacement response at a nodal point A is displayed in the upper right of Fig. 1 as a function of time, as evaluated for \( n = 3 \) terms in the frequency power series expansion of Section 2, and compared with results obtained from the analysis of the drumhead as a whole, given as a series of Bessel functions [8]. Improved results may be achieved by considering additional terms in the frequency power series [7].

Figure 1: Patch cut out from a drumhead and then submitted to initial velocity \( v_0 \) and to time-dependent, equivalent nodal forces along the boundary (see text).

3.2 Sudden gravity acceleration applied to an irregular-shaped body

The fixed-free elastic body (rectangular, wide bar element) in the upper left of Fig. 2 is submitted to sudden gravity acceleration, for homogeneous initial conditions. Owing to the one-dimensional character of the solicitation (Poisson’s ratio equal to zero), this problem can be solved in the frame of the theory of potential. A particular solution of the equilibrium equation \( \sigma_{rr} - \rho \ddot{u}_r - \rho g = 0 \) is \( \sigma_r^b = -\rho g (\ell - y) \), leading to \( u_r^b = -\rho g (\ell - y)^2 / 2E \), in which \( \ell \) is the bar length, \( g \) is the acceleration of gravity and \( E \) is the elasticity modulus. An irregular-shaped, elastic body is cut out of the bar element and subsequently submitted to
both time-dependent tractions along the free boundary edges, as evaluated from the bar's analytical response along the cutting lines, and body forces [6]. Although both the underlying analytical problem and the adopted particular solution are extremely simple, the present patch test is a rigorous one, as the simple dynamic action becomes a very complicated set of equivalent, time-dependent nodal loads applied to the irregular body. The numerical model consists of a total of 59 linear, equally spaced boundary elements, for \( n = 3 \) terms in the frequency power series expansion. The displacement response at point \( A(10, 15) \) is displayed in the lower right of Fig. 2 as a function of time. The remaining two graphics display displacement values along the whole set of boundary nodal points as well as along the depicted line segment, at given time instants. One may observe how the corners of the cutout negatively affect the results, indicating the need of a more refined mesh. Target results are given by the Fourier series solution of the proposed fixed-free bar problem.

![Image of patch test](image)

Figure 2: Patch test for a fixed-free bar submitted to sudden gravity acceleration. Displacements are displayed at a nodal point, as a function of time, as well as both at nodal points and along an internal line segment, for some time instants.

**Conclusions**

The frequency-domain formulation of the hybrid boundary element method was presented for the analysis of general time-dependent problems. The power series expansion in frequencies ensures that the dynamic differential equilibrium
equation is accurately satisfied in the domain, although no internal points or cells are used, as in the conventional boundary element method. It is also worth mentioning that the whole formulation is based on real fundamental functions. Moreover, results at internal points are given directly as a superposition of interpolating functions, which makes post-processing an easy task. Accuracy is as high as, possibly higher than in other boundary element methods. This generalized frequency-domain formulation and the outlines concerning non-homogeneous initial conditions, forced nodal displacements and prescribed domain forces apply to finite elements as well. Owing to space restrictions, numerical results had to be presented very succinctly. However, their quality is evident, since the displayed patch tests are rather demanding, not only for their irregular shapes but mainly because of the considered cases of non-homogeneous initial conditions and gravity forces.

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References