DRBEM solution of temperatures and velocities in DC cast aluminium slabs

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Abstract

The steady-state temperature and velocity field problem associated with direct-chill, semi-continuously cast aluminium slabs has been solved using the dual reciprocity boundary element method (DRBEM). The solution is based on formulation which incorporates the one-phase physical model, general DRBEM transport equation solution strategy, Laplace equation fundamental solution weighting, and augmented thin plate splines for transforming the domain integrals into a finite series of boundary integrals. Realistic non-linear boundary conditions and temperature variation of all material properties are included. Two-dimensional test case solutions are shown.

1 Introduction

Direct-chill (DC) casting is currently the most common semi-continuous casting practice in production of aluminium alloys. The process involves molten metal being fed through a bottomless water-cooled mould where it is sufficiently solidified around the outer surface that it takes the shape of the mould and acquires sufficient mechanical strength to contain the molten core at the centre. As the strand emerges from the mould, water impinges directly from the mould onto the surface (direct-chill), falls over the cast surface and completes the solidification. A schematic of the process is given in Figure 1. Related transport, solid-mechanics, and phase change kinetics phenomena are extensively studied [1] and many different BEM based numerical methods [2] have been used to solve the respective DC casting models in the past. The proper prediction of the temperature, velocity, species, and phase fields in the strand is one of the prerequisites for automatization of the process and related optimization with respect to its quality and productivity. The IMPOL Slovenska Bistrica Aluminium Industry started to
support the computational modelling of the process already a decade ago. A DRBEM based [3] temperature field model [4] was used in automatization algorithms of the process. This relatively simple model has been extensively verified against experimental data [5, 6]. This paper represents a coupled thermal and fluid flow model of the process, based on solution strategy from [7].

![Figure 1: Schematics of the DC casting process.](image)

### 2 Governing equations

The coupled mass, momentum, and energy transfer in DC casting can be reasonably represented in the framework of the one-phase continuum formulation [8] which assumes local thermal equilibrium between the phases. This formulation can in solidification context involve quite complicated constitutive equations. However, because of the conference paper limitations, these have been introduced here in its most simplified form in order to point out the computational methodology instead of the physics. Consider a connected fixed domain \( \Omega \) with boundary \( \Gamma \) occupied by a phase change material described with the temperature dependent density \( \rho_P \) of the phase \( P \), temperature dependent specific heat at constant pressure \( c_p \), effective thermal conductivity \( k \), viscosity \( \mu \) and the specific latent heat of the solid-liquid phase change \( h_M \). The one-phase continuum formulation of the transport equations for the assumed system is

\[
\frac{\partial}{\partial t} \rho + \nabla \cdot (\rho \mathbf{v}) = 0 \tag{1}
\]

\[
\frac{\partial}{\partial t} (\rho h) + \nabla \cdot (\rho \mathbf{v} h) = \nabla \cdot (k \nabla T) \tag{2}
\]

\[
\frac{\partial}{\partial t} (\rho \mathbf{v}) + \nabla \cdot (\rho \mathbf{v} \mathbf{v}) = -\nabla P + \nabla \cdot (\mu \nabla \mathbf{v}) + \mathbf{f} \tag{3}
\]
with mixture density defined as \( \rho = f_S \rho_S + f_L \rho_L \), mixture enthalpy defined as \( h = ( f_S \rho_S h_S + f_L \rho_L h_L ) / \rho \), mixture velocity defined as \( \mathbf{v} = ( f_S \rho_S \mathbf{v}_S + f_L \rho_L \mathbf{v}_L ) / \rho \) and mixture enthalpy defined as \( h = ( f_S \rho_S h_S + f_L \rho_L h_L ) / \rho \). \( P \) stands for mixture pressure. The constitutive mixture temperature \( T \) - mixture enthalpy relationships are

\[
h_S = \int_{T_{ref}}^{T} c_p dT, \quad h_L = \int_{T_{ref}}^{T} c_p dT + h_M
\]

Thermal conductivity and viscosity can arbitrarily depend on temperature. The variation of the density with temperature is included in the body force term by using the Boussinesq approximation

\[
f = \rho \mathbf{a} (1 - \beta (T - T_{ref})) - \rho \mathbf{P} (1 - f_L) \mathbf{v}
\]

with \( \mathbf{a} \) standing for the acceleration vector, \( \beta \) for the volumetric thermal expansion coefficient, and \( T_{ref} \) for the reference temperature. The second term in the body force is modelling the phase change, simply constituted through the phase change coefficient \( c_P \) (large number) and temperature dependent liquid fraction \( f_L \) which is assumed to vary as

\[
f_L(T) = \begin{cases} 0 & T < T_S \\ (T - T_S)/(T_L - T_S) & T_S \leq T \leq T_L \\ 1 & T > T_L \end{cases}
\]

where \( T_S \) stands for the solidus and \( T_L \) for the liquidus temperature. We seek for mixture temperature, velocity and pressure at time \( t_0 + \Delta t \) by assuming known temperature and velocity fields at time \( t_0 \) and boundary conditions of the Dirichlet, Neumann and Robin type, respectively.

### 3 Solution procedure

The solution of the problem posed is based on the general transport equation defined on a fixed domain \( \Omega \) with boundary \( \Gamma \), standing for a reasonably broad spectra of mass, energy, momentum, and species transfer problems

\[
\frac{\partial}{\partial t} [ \rho \mathcal{C}(\Phi) ] + \nabla \cdot [ \rho \mathbf{v} \mathcal{C}(\Phi) ] = -\nabla \cdot (-\mathbf{D} \nabla \Phi) + S
\]

with \( \rho, \mathcal{C}, t, \mathbf{v}, \mathbf{D}, \) and \( S \) standing for density, transport variable, time, velocity, diffusion matrix and source, respectively. Scalar function \( \mathcal{C} \) stands for possible more involved constitutive relations. The solution of the governing equation for the transport variable at the final time \( t = t_0 + \Delta t \) is sought, where \( t_0 \) represents the initial time and \( \Delta t \) the positive time increment. The solution is constructed by the initial and boundary conditions that follow. The initial value of transport variable \( \Phi(p, t_0) \) at a point with position vector \( p \) and time \( t_0 \) is defined through the known function \( \Phi_0 \)

\[
\Phi(p, t) = \Phi_0; \quad p \in \Omega \cup \Gamma
\]
The boundary is divided into not necessarily connected parts \( \Gamma = \Gamma^D \cup \Gamma^N \cup \Gamma^R \) with Dirichlet, Neumann and Robin type boundary conditions respectively. These boundary conditions are at the boundary point \( \mathbf{p} \) with normal \( \mathbf{n}_\Gamma \) and time \( t_0 \leq t \leq t_0 + \Delta t \) defined through known functions \( \Phi^D_\Gamma, \Phi^N_\Gamma, \Phi^R_\Gamma \) and \( \Phi^{R, \text{ref}}_\Gamma \)

\[
\Phi(\mathbf{p}, t) = \Phi^D_\Gamma; \quad \mathbf{p} \in \Gamma^D, \quad \frac{\partial}{\partial n_\Gamma} \Phi(\mathbf{p}, t) = \Phi^N_\Gamma; \quad \mathbf{p} \in \Gamma^N
\]

\[
\frac{\partial}{\partial n_\Gamma} \Phi(\mathbf{p}, t) = \Phi^R_\Gamma (\Phi - \Phi^{R, \text{ref}}_\Gamma); \quad \mathbf{p} \in \Gamma^R
\]  

(8)

The involved parameters of the governing equation and boundary conditions are assumed to depend on the transport variable, space and time. The solution procedure thus inherently invokes iterations. The governing equation is transformed as follows. The diffusion matrix

\[
D = D \mathbf{I} + D'
\]  

(9)

is split into constant isotropic part \( D \mathbf{I} \), with \( \mathbf{I} \) denoting identity matrix, and the remaining nonlinear anisotropic part \( D' \). The transport equation is subsequently cast into Poisson form

\[
\nabla^2 \Phi = \nabla \cdot \Theta + \theta
\]

\[
\theta = [\frac{\partial}{\partial t} (\rho C(\Phi)) - S] / D, \quad \Theta = [\rho \mathbf{v} C(\Phi) - D' \nabla \Phi] / D
\]  

(10)

The inhomogenous terms are expanded as

\[
\theta \approx \hat{\theta} + \hat{\theta}, \Phi (\Phi - \hat{\Phi}), \quad \Theta \approx \hat{\Theta} + \hat{\Theta}, \Phi (\Phi - \hat{\Phi})
\]  

(11)

with ‘hat’ denoting value at previous iteration. The final form of the transformed equation is

\[
[\nabla^2 - \nabla \cdot \hat{\Theta}, \Phi - \hat{\theta}, \Phi] \Phi = \nabla \cdot \hat{\Theta} - \nabla \cdot \hat{\Theta}, \Phi \hat{\Phi} + \hat{\theta} - \hat{\theta}, \Phi \hat{\Phi}
\]

The transformed governing equation is time-discretized in a fully implicit manner, i.e.

\[
\frac{\partial}{\partial t} (\rho C(\Phi)) \approx \frac{\rho C(\Phi) - \rho_0 C(\Phi_0)}{\Delta t}
\]

Subscript 0 denotes value at the initial time \( t = t_0 \) and no subscript denotes value at time \( t_0 + \Delta t \). The governing equation is space-discretized by weighting the time-discretized governing equation over the domain \( \Omega \) by the fundamental solution of the Laplace equation \( T^* \). Let us focus on two-dimensional (described in Cartesian coordinates \( p_x, p_y \) with base vectors \( \mathbf{i}_x, \mathbf{i}_y \)) situations

\[
T^*(\mathbf{p}; s) = \frac{1}{2 \pi} \log \frac{r_0}{r}
\]  

(12)
with $r^2 = (p_x - s_x)^2 + (p_y - s_y)^2$ and $r_0$ standing for the scaling constant. The following two integral equations can be deduced by assuming the properties of the fundamental solution and the Green's theorems

$$\int_{\Gamma} T^* \frac{\partial \Phi}{\partial n_\Gamma} d\Gamma - \int_{\Gamma} \frac{\partial T^*}{\partial n_\Gamma} \Phi d\Gamma - c^*_s \Phi_s = \int_{\Omega} [\nabla \cdot \Theta + \theta] T^* d\Omega$$

(13)

$$\int_{\Gamma} \nabla T^* \frac{\partial \Phi}{\partial n_\Gamma} d\Gamma - \int_{\Gamma} \nabla \frac{\partial T^*}{\partial n_\Gamma} \Phi d\Gamma + \nabla (c^*_s \Phi_s) = \int_{\Omega} [\nabla \cdot \Theta + \theta] \nabla T^* d\Omega$$

(14)

with $c^*(s)$ standing for the fundamental solution related coefficient. The right hand sides of the integral equations (13,14) are, for shorting the notation, written back in unexpanded form. The integral equation (13) is used for calculation of the unknown $\Phi$ and the integral equation (14) is used for calculation of the partial derivatives of $\Phi$: $\Phi, \zeta; \zeta = x, y$. The integral types that arise from defined weighting can be classified into the following six types

$$\int_{\Omega} \nabla^2 \mathcal{F} T^* d\Omega, \quad \int_{\Omega} \mathcal{F} T^* d\Omega, \quad \int_{\Omega} \nabla \cdot \mathcal{G} T^* d\Omega$$

(15)

$$\int_{\Omega} \nabla^2 \mathcal{F} \nabla T^* d\Omega, \quad \int_{\Omega} \mathcal{F} \nabla T^* d\Omega, \quad \int_{\Omega} \nabla \cdot \mathcal{G} \nabla T^* d\Omega$$

(16)

where $\mathcal{F}$ and $\mathcal{G}$ refer to scalar and vector valued functions. They are computed as follows. Boundary geometry is approximated by $N_\Gamma$ straight line segments, and spatial variation of the fields on each of the boundary segments is represented by constant interpolation functions with gridpoints coinciding with the geometrical centers of the straight line segments. This simplest discretization assumption is made here for clarity of the presentation only. Higher order space and time approximations can of course be introduced. The Einstein summation convention is used in this text, i.e. any index which is repeated twice in a product is summed up. An underlined index is not summed up. For boundary integrals of a scalar valued function, or boundary integrals of a component of a vector valued function, both denoted with function $\mathcal{F}$, the first integral can be computed as

$$\int_{\Omega} \nabla^2 \mathcal{F} T_i^* d\Omega \approx G_{lk} \delta_{ki} \frac{\partial \mathcal{F}_i}{\partial n_\Gamma} - H_{lk} \delta_{ki} \mathcal{F}_i - c^*_i \delta_{li} \mathcal{F}_i$$

(17)

where $k = 1, 2, \ldots N_\Gamma$ and $i, l = 1, 2, \ldots N$. $N = N_\Gamma + N_\Omega$ is the total number of points in which the solution is calculated. The first $N_\Gamma$ points $\mathbf{p}_n$ coincide with the boundary gridpoints and the last $N_\Omega$ points are arbitrarily distributed over the domain $\Omega$. Index $l$ stands for $\mathbf{s}_l = \mathbf{p}_l$. $\delta$ represents the Kronecker symbol. Matrix elements $G_{lk}$ and $H_{lk}$ are defined as follows

$$G_{lk} = \int_{\Gamma_k} T_i^* d\Gamma, \quad H_{lk} = \int_{\Gamma_k} \frac{\partial T_i^*}{\partial n_\Gamma} d\Gamma,$$
where $\Gamma_k$ represents the $k$-th boundary segment, and $c_i^*$ is equal to

$$c_i^* = 0; \quad s_i \not\in \Omega \cup \Gamma, \quad c_i^* = \frac{1}{2}; \quad s_i \in \Gamma, \quad c_i^* = 1; \quad s_i \in \Omega. \quad (19)$$

The related two integrals are calculated as

$$\int_{\Omega} \nabla^2 F \frac{\partial T_i^*}{\partial \xi} \, d\Omega \approx G_{lk, \xi} \delta_{ki} \frac{\partial \mathcal{F}_i}{\partial n_\Gamma} - H_{lk, \xi} \delta_{ki} \mathcal{F}_i + c_i^* \delta_{li} \frac{\partial \mathcal{F}_i}{\partial p_\xi} \quad (20)$$

with $\xi$ standing for $x$ and $y$ and

$$G_{lk, \xi} = \int_{\Gamma_k} \frac{\partial T_i^*}{\partial p_\xi} \, d\Gamma, \quad H_{lk, \xi} = \int_{\Gamma_k} \frac{\partial^2 T_i^*}{\partial p_\xi \partial n_\Gamma} \, d\Gamma, \quad (21)$$

The remaining four domain integrals are transformed by considering the approximation of the spatial variation of the fields in $\Omega$ by the global interpolation functions of the form

$$\mathcal{F}(p) \approx \psi_u(p) \varsigma_u; \quad u = 1, 2, \cdots, N + 3$$

The two-dimensional scaled augmented thin plate splines are used in this work

$$\psi_n(p) = r_n^2 \log r_n; \quad n = 1, 2, \cdots, N$$

$$\psi_{N+1}(p) = p_x - p_x^0, \quad \psi_{N+2}(p) = p_y - p_y^0, \quad \psi_{N+3}(p) = 1 \quad (22)$$

with

$$r_n^2 = (p - p_n) \cdot (p - p_n) \quad (23)$$

Scaling constants $p_x^0$ and $p_y^0$ stand for the mean coordinates of the domain $\Omega$. Coefficients $\varsigma_u$ are calculated by constructing a system of $N + 3$ algebraic equations

$$\Psi \varsigma = \mathcal{F} \quad (24)$$

The vectors are $\varsigma = (\varsigma_1, \varsigma_2, \cdots, \varsigma_{N+3})^T$ and $\mathcal{F} = (\mathcal{F}_1, \mathcal{F}_2, \cdots, \mathcal{F}_N, 0, 0, 0)^T$. The first $v = 1, 2, \cdots, N$ rows of matrix $\Psi$ are of the form $(\psi_{v1}, \psi_{v2}, \cdots, \psi_{vN+3})$, and the last three rows $v = N + 1, N + 2, N + 3$ are of the form $(\psi_{1v}, \psi_{2v}, \cdots, \psi_{Nv}, 0, 0, 0)$, where the notation has been shortened to $\mathcal{F}_n \equiv \mathcal{F}(p_n)$, $\psi_{nu} \equiv \psi_u(p_n)$. Coefficients $\varsigma_u$ follow by inverting the system (24)

$$\varsigma = \Psi^{-1} \mathcal{F} \quad (25)$$

Consequently, the second domain integral can be written in a compact dual reciprocity form ($k = 1, 2, \cdots, N_\Gamma, \quad i, l = 1, 2, \cdots, N, \quad u = 1, 2, \cdots, N + 3$)

$$\int_{\Omega} \mathcal{F}_i T_i^* \, d\Omega \approx \int_{\Omega} \psi_{ui} \Psi_{ui}^{-1} \mathcal{F}_i T_i^* \, d\Omega = \hat{\psi}_{lu} \Psi_{ui}^{-1} \mathcal{F}_i, \quad (26)$$

with

$$\hat{\psi}_{lu} \equiv \int_{\Omega} \psi_u T_i^* \, d\Omega \quad (27)$$
The integral \( \hat{\Psi}_{lu} \) is calculated by defining the harmonic functions \( \hat{\psi}_u \)

\[
\nabla^2 \hat{\psi}_u(p) = \psi_u(p)
\]

(28)

which allow us to write

\[
\hat{\Psi}_{lu} = \int_{\Gamma} \frac{\partial \hat{\psi}_u}{\partial n_{\Gamma}} T^* d\Gamma - \int_{\Gamma} \hat{\psi}_u \frac{\partial T^*}{\partial n_{\Gamma}} d\Gamma - c^*_l \hat{\psi}_u(s)
\]

(29)

The upper integrals are numerically evaluated by using the same discretization strategy that leads to expression (17)

\[
\int_{\Omega} \nabla^2 \hat{\psi}_u T^*_l d\Omega \approx G^\psi_{lk} \delta_{ki} \frac{\partial \hat{\psi}_{lu}}{\partial n_{\Gamma}} - H^\psi_{lk} \delta_{ki} \hat{\psi}_{iu} - c^*_l \delta_{li} \hat{\psi}_{iu}
\]

(30)

However, matrices \( G^\psi \) and \( H^\psi \) could differ from matrices \( G \) and \( H \) because the boundary subdivision that corresponds to the calculation of the fields on the boundary and integrals \( \hat{\Psi}_{lu} \) could differ in general. Let us denote the number of boundary segments leading to calculation of matrices \( G^\psi \) and \( H^\psi \) with \( N_{\Gamma}^\psi \). Therefore index \( k \) in equation (30) runs as \( k = 1, 2, \ldots, N_{\Gamma}^\psi \).

The adjacent harmonic functions \( \hat{\psi}_u \) for the thin plate splines (22) used (the selection is not unique!) in the present work are

\[
\hat{\psi}_n = \frac{1}{16} r_n^4 \log r_n - \frac{1}{32} r_n^4, \quad \hat{\psi}_{N+1} = \frac{1}{6} (p_x - p_x^0)^3
\]

\[
\hat{\psi}_{N+2} = \frac{1}{6} (p_y - p_y^0)^3, \quad \hat{\psi}_{N+3} = \frac{1}{4} (p_x - p_x^0)^2 + \frac{1}{4} (p_y - p_y^0)^2
\]

(31)

The related two domain integrals are calculated as

\[
\int_{\Omega} \mathcal{F} \frac{\partial T^*_l}{\partial p_{\xi}} d\Omega \approx \hat{\Psi}_{lu,\xi} \hat{\Psi}_{ui}^{-1} \mathcal{F}_i; \quad \hat{\Psi}_{lu,\xi} \equiv \int_{\Omega} \psi_u \frac{\partial}{\partial p_{\xi}} T^*_l d\Omega
\]

(32)

The upper integrals are evaluated similarly to those in equation (30)

\[
\hat{\Psi}_{lu,\xi} \approx G^\psi_{lk,\xi} \delta_{ki} \frac{\partial \hat{\psi}_{lu}}{\partial n_{\Gamma}} - H^\psi_{lk,\xi} \delta_{ki} \hat{\psi}_{iu} - c^*_l \delta_{li} \frac{\partial \hat{\psi}_{iu}}{\partial p_{\xi}}
\]

(33)

with index \( k \) running as \( k = 1, 2, \ldots, N_{\Gamma}^\psi \). The third integral is evaluated as

\[
\int_{\Omega} \nabla \cdot \mathcal{G} T^*_l d\Omega = \hat{\Psi}_{lu,z} \Psi_{ui}^{-1} \mathcal{G}_{x i} + \hat{\Psi}_{lu,y} \Psi_{ui}^{-1} \mathcal{G}_{y i}
\]

(34)

and the related two domain integrals are calculated as

\[
\int_{\Omega} \nabla \cdot \mathcal{G} \frac{\partial T^*_l}{\partial p_{\xi}} d\Omega = \hat{\Psi}_{lu,x\xi} \Psi_{ui}^{-1} \mathcal{G}_{x i} + \hat{\Psi}_{lu,y\xi} \Psi_{ui}^{-1} \mathcal{G}_{y i}
\]

(35)
\[ \Psi_{lu,\varphi} \equiv \int_{\Omega} \frac{\partial \psi_u}{\partial p} \frac{\partial T^*_v}{\partial p} \, d\Omega \]  
(36)

\[ \Psi_{lu,\varphi} \approx G_{lk,\varphi} \delta_{ki} \frac{\partial^2 \hat{\psi}_{iu}}{\partial p \partial \rho} - H_{lk,\varphi} \delta_{ki} \frac{\partial \hat{\psi}_{iu}}{\partial p} + c_{li} \delta_{li} \frac{\partial^2 \hat{\psi}_{iu}}{\partial p \partial \rho} \]  
(37)

After each solution of the scalar \( \Phi \) a relaxation with coefficient \( \epsilon_{rel} \) is made \( \Phi = \hat{\Phi} + \epsilon_{rel} (\Phi - \hat{\Phi}) \) and the timestep iterations are stopped when the criterion \( (|\Phi_{avg} - |\Phi_{avg}|)/\Phi_{avg} < \epsilon_{itr} \) is reached, where subscript avg represents the average gridpoint value. The criterion of reaching the steady state is \( (|\Phi_{avg} - |\Phi_{avg,0}|)/|\Phi_{avg,0}| < \epsilon_{sta} \). The three governing equations (pressure correction Poisson equation replaces the mass conservation equation) are in each timestep solved in an iterative bundle. Their representation within general transport equation context is shown in Appendix. First, new pressure field is solved, based on the old velocity, pressure and temperature fields. Momentum equation is solved afterwards, based on the old temperature field and new pressure field. Subsequently, pressure correction field is solved based on the new velocity field. The new velocity field is corrected through the pressure correction field. Finally new temperature field is calculated, based on the new velocity field.

4 Numerical implementation and numerical examples

The elements of the involved boundary integral matrices and their derivatives are evaluated analytically. The numerical implementation of the code involves linear and quadratic elements as well. The corresponding formulas will appear elsewhere. The application of the discretization and the described boundary conditions gives square system of \( N \) linear equations for solving \( \Phi \) in the domain points and \( \Phi \) or \( \partial \Phi/\partial \Gamma \) in the boundary points. The involved systems of algebraic equations are solved, except for the pressure and pressure correction equations, by using GMRES technique with Jacobi preconditioning [12]. In case of the two pressure equations which are singular due to the prescribed Neumann boundary conditions over the whole boundary \( \Gamma \), the systems are solved by the Householder reduction to bidiagonal form and QR diagonalization with shifts.

Two-dimensional test cases are considered as shown in Figure 2. Because of the conference paper limitations, and at the same time possible reproduction of the results, constant material properties are used, chosen in material properties range of aluminium alloys \( \rho = 2800 \text{kg/m}^3, c_p = 1000 \text{J/(kgK)}, k = 100 \text{W/(mK)}, T_S = T_{ref} = 850 \text{K}, T_C = 900 \text{K}, h_M = 300000 \text{J/kg}, \mu = 0.012 \text{Ns/m}^2, \beta = 2.5^{-6} \text{K}^{-1}, c_P = 10^5 \text{kg m}^{-3} \text{s}^{-1} \). It should be emphasized that the formulation permits arbitrary realistic material properties. The temperature, pressure and velocity is calculated on the two-dimensional domain, defined with the coordinates \( 0.0 \text{m} \leq p_x \leq 0.1 \text{m} \).
0.0 m ≤ \( p_y \) ≤ -1.0 m. North (N) and south (S) boundaries have Dirichlet boundary conditions with \( T_{TN} = 950 \) K, South boundary has Dirichlet boundary conditions \( T_{TS} = 300 \) K. The part of the east (E) boundary \( 0.0 \) m ≤ \( p_y \) ≤ -0.1 m, which coincides with the mould (see schematic dark vertical rectangles in Figure 2), is subject to Neumann boundary condition with \( T_{TE} = 100 \) 000 W/m², and the direct-chill part of the east boundary -0.1 m < \( p_y \) ≤ -1.0 m is subject to the Robin boundary condition with \( T_{TrefE} = 300 \) K, \( H_{TE} = 5000 \) W/(m² K). The west (W) part of the boundary is insulated \( F'_{TW} = 0 \) W/m² (symmetry condition). The assumed initial temperature distribution is linear between \( T_{TN} \) and \( T_{TS} \), and the velocity field is set to the casting velocity 0.000833 m/s. Fixed timestep 1 s is used. All relaxation factors have been set to \( c_{rel} = 0.01 \), all iteration margins to \( \epsilon_{itr} = 0.0001 \), and the steady state criterions to \( \epsilon_{sts} = 0.001 \). The non-uniform mesh with 76 boundary elements and 229 domain nodes is shown in Figure 2. The test cases are reaching steady state in approximately 1000 timesteps. The CPU time for one test case is around two hours on Pentium II-400 Mhz computer with Windows NT operating system and Digital Visual Fortran compiler. The two-dimensional results have been compared with the results obtained by the finite volume method. Excellent agreement regarding the interphase region position and overall velocity and temperature distribution has been found. This comparison will be given in a forthcoming journal paper.

5 Conclusions

Present paper demonstrates successful use of the DRBEM for numerical evaluation of physical model that could be previously solved by more established numerical methods only. It probably represents the first industrial use of the boundary element method for solving coupled transport phenomena of such complexity. The essentially new general transport phenomena framework for the DRBEM solution is applied. The framework is sufficiently broad for inclusion of the completely anisotropic, non-linear, and multiphase problems. Its main characteristics is in an unified form of the solution procedure, able to cope with complicated coupled transport phenomena in an ordered way. The described features give DRBEM similar flexibility for coping with the complicated constitutive relations like classical numerical approaches such as the finite volume method. Present research is focused on upgrades of the method to three-dimensions, axisymmetry, and coping of binary systems (species transfer). The described computational model found application in DC casting simulator of aluminium alloys in IMPOL Slovenska Bistrica where it has been coupled with the realistic material properties, mould heat transfer correlations, and direct-chill boundary conditions with provisions for one phase forced convection, subcooled nucleate boiling, film boiling, immersion in stagnant pool, and radiation.
Figure 2: Coarse mesh solution of the temperature and velocity fields. Left: melt flow is restricted over the medium part of the upper surface. Right: melt flow is restricted over the central part of the upper surface. The difference between the two isotherms ($T_S$ and $T_C$ are bold) is 25 K.

Acknowledgement

The authors would like to acknowledge the aluminium enterprise IMPOL Slovenska Bistrica (http://www.impol.com), Slovenia and the Ministry for Science and Technology of the Republic of Slovenia (http://www.mzt.si) for support in the framework of the projects Optimization of the Continuous Casting Process and COST-P3: Simulation of Physical Phenomena in Technological Applications that form a part of the Slovenian national research programme Multiphase Systems.
References


Appendix: Definition of coefficients

N×N systems of equations

Energy transport

variable definition
Φ \[ T \]
\[ C(\Phi) \] \[ h(T) \]
D \[ k \]
D_{\xi} \[ 0 \]
S \[ 0 \]
\[ \theta = g C(\Phi)/(\Delta t \ k) - g C(\Phi_0)/(\Delta t \ k) \]
\[ \theta_{,x} = g (c_p \Phi + h_M f_c, \Phi)/(\Delta t \ k) \]
\[ \Theta_x \] \[ g v_x C(\Phi) \]
\[ \Theta_y \] \[ g v_y C(\Phi) \]
\[ \Theta_{x,\Phi} \] \[ g v_x (c_p \Phi + h_M f_c, \Phi) \]
\[ \Theta_{y,\Phi} \] \[ g v_y (c_p \Phi + h_M f_c, \Phi) \]

X-Momentum transport

variable definition
Φ \[ v_x \]
\[ C(\Phi) \] \[ v_x \]
D \[ \mu \]
D_{\xi} \[ 0 \]
S \[ \] \[ -P_{,x} + g a_x (1 - \beta(T - T_{ref})) - c_p (1 - f_c) \Phi \]
\[ \theta = g \Phi/(\Delta t \mu) - g \Phi_0/(\Delta t \mu) - S/\mu \]
\[ \theta_{,x} = g/(\Delta t \mu) + c_p (1 - f_c)/\mu \]
\[ \Theta_x \] \[ g \Phi^2/\mu \]
\[ \Theta_y \] \[ g v_y \Phi/\mu \]
\[ \Theta_{x,\Phi} \] \[ 2 g \Phi/\mu \]
\[ \Theta_{y,\Phi} \] \[ g v_y/\mu \]

Y-Momentum transport

variable definition
Φ \[ v_y \]
\[ C(\Phi) \] \[ v_y \]
D \[ \mu \]
D_{\xi} \[ 0 \]
S \[ \] \[ -P_{,y} + g a_y (1 - \beta(T - T_{ref})) - c_p (1 - f_c) \Phi \]
\[ \theta = g \Phi/(\Delta t \mu) - g \Phi_0/(\Delta t \mu) - S/\mu \]
\[ \theta_{,x} = g/(\Delta t \mu) + c_p (1 - f_c)/\mu \]
\[ \Theta_x \] \[ g v_x \Phi/\mu \]
\[ \Theta_y \] \[ g \Phi^2/\mu \]
\[ \Theta_{x,\Phi} \] \[ g v_x/\mu \]
\[ \Theta_{y,\Phi} \] \[ 2 g \Phi/\mu \]
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\( \mathbf{N}_T \times \mathbf{N}_T \) systems of equations

**Pressure Poisson equation**

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<th>variable</th>
<th>definition</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \Phi )</td>
<td>( P )</td>
</tr>
<tr>
<td>( \mathcal{C}(\Phi) )</td>
<td>( P )</td>
</tr>
<tr>
<td>( D )</td>
<td>1</td>
</tr>
<tr>
<td>( D' )</td>
<td>0</td>
</tr>
<tr>
<td>( S )</td>
<td>0</td>
</tr>
<tr>
<td>( \theta )</td>
<td>0</td>
</tr>
<tr>
<td>( \theta', \phi )</td>
<td>0</td>
</tr>
</tbody>
</table>

\[
\begin{align*}
\Theta_x &= -\rho \frac{v_x}{\Delta t} + \rho v_0 \frac{z_0}{\Delta t} - \rho \left( 2 v_x v_{z,x} + v_x v_{y,y} + v_y v_{z,y} \right) \\
&\quad + \mu \left( v_{x,x} + v_{z,y,y} \right) + \rho a_x \left( 1 - \beta \left( T - T_{ref} \right) \right) - c_p \left( 1 - f_e \right) v_x \\
\Theta_y &= -\rho \frac{v_y}{\Delta t} + \rho v_0 \frac{z_0}{\Delta t} - \rho \left( 2 v_y v_{y,y} + v_x v_{z,y} + v_z v_{y,x} \right) \\
&\quad + \mu \left( v_{y,y} + v_{z,x,x} \right) + \rho a_y \left( 1 - \beta \left( T - T_{ref} \right) \right) - c_p \left( 1 - f_e \right) v_y \\
\Theta_{x,\phi} &= 0 \\
\Theta_{y,\phi} &= 0
\end{align*}
\]

Pressure Poisson equation Neumann boundary conditions

\[
\frac{\partial P}{\partial n_T} = \left[ -\frac{\partial}{\partial t} \left( \rho v \right) - \nabla \cdot \left( \rho v v \right) + \nabla \cdot \left( \mu \nabla v \right) + f \right] \cdot n_T
\]

**Pressure correction Poisson equation**

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<tr>
<td>( D' )</td>
<td>0</td>
</tr>
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<td>( S )</td>
<td>0</td>
</tr>
<tr>
<td>( \theta )</td>
<td>( -(\rho - \rho_0)/\Delta t )</td>
</tr>
<tr>
<td>( \theta, \phi )</td>
<td>0</td>
</tr>
<tr>
<td>( \Theta_x )</td>
<td>( -\rho v_x )</td>
</tr>
<tr>
<td>( \Theta_y )</td>
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</tr>
<tr>
<td>( \Theta_{x,\phi} )</td>
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</tr>
<tr>
<td>( \Theta_{y,\phi} )</td>
<td>0</td>
</tr>
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Pressure correction Poisson equation Neumann boundary conditions

\[
\frac{\partial P'}{\partial n_T} = 0
\]