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Accurate computation of Green's functions in regions of complex configuration

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Abstract

Special integral representations (referred herein to as *modified potentials*) are proposed to accurately compute Green's functions for boundary value problems stated for Laplace and biharmonic equation over regions of complex configuration. Kernels of the potentials are built with the aid of Green's functions for simply shaped regions. The emphasis is also on the modified potentials, whose observation and source points occupy different sets. Such potentials reduce the original boundary value problems to functional (integral) equations with smooth kernels.

1 Potential problems

Kernels of the classical elliptic potentials are composed of fundamental solutions of governing differential equations. In many practical settings, however, the use of Green's functions is preferable. The method of modified potentials (MMP) resulting from that idea, has successfully been applied in Melnikov [6,8,10] to a number of boundary value problems.

The range of successful applications of the MMP is not that broad compared to the method of classical potentials (MCP) (see Lovitt [1], Smirnov [3], Kupradze [4], Brebbia [7], and Golberg and Chen [9]). However, when applicable, MMP is notably more efficient, because, when using this method, there is no need to numerically satisfy all the boundary conditions imposed.

Computational algorithms based on the MMP are compared in this study against the traditional boundary integral equation method based routines. Test examples for Laplace and biharmonic equation were considered.

Let $G(z,\zeta)$ represent the Green's function of the Dirichlet problem

$$\Delta u(z)=0, \quad z\in\Omega$$

$$u(z)=0, \quad z\in\Lambda$$

posed over a simply-connected region Ω bounded with a smooth curve Λ.

We decompose $G(z,\zeta)$ as follows

$$G(z,\zeta)=\frac{1}{2\pi}\ln\frac{1}{|z-\zeta|}+g(z,\zeta).$$

Since $G(z,\zeta)$ vanishes on Λ by definition, its regular component $g(z,\zeta)$, being considered as a function of z, has to be a solution to the problem

$$\Delta g(z,\zeta^*)=0, \quad z\in\Omega \tag{1}$$

$$g(z,\zeta^*)=-T(z,\zeta^*), \quad z\in\Lambda \tag{2}$$

where ζ^* is an arbitrarily fixed point on Ω, while $T(z,\zeta^*)$ represents the trace of the singular logarithmic component of $G(z,\zeta^*)$ on Λ.

1.1 Double-layer potential approach

In accordance with the standard [1, 3] MCP routine, the solution to the problem (1)-(2) is written as

$$g(z,\zeta^*)=\int_\Lambda \frac{\partial}{\partial\nu}\left(\ln\frac{1}{|z-t|}\right) q(t)d_t\Lambda, \quad z\in\Omega,\ t\in\Lambda \tag{3}$$

where ν is a normal direction to Λ at the point t. The density $q(t)$ can be determined while satisfying the boundary condition of eqn (2). By virtue of the jump-property of potential (3), one arrives at the following Fredholm integral equation

$$-\pi q(z)+\int_\Lambda \frac{\partial}{\partial\nu}\left(\ln\frac{1}{|z-t|}\right) q(t)d_t\Lambda=-T(z,\zeta^*), \quad z\in\Lambda \tag{4}$$

in $q(t)$. The kernel of the above equation is a continuous function. Hence its numerical solution is quite trivial.

Thus, the double-layer potential version of the MCP seemingly provides a solid well-posed procedure for computing Green's functions. Indeed, the original two-dimensional boundary value problem reduces to a regular integral equation. However, the practical computation comes to a conflict with such an expectation, revealing a notable shortcoming in the method. In what follows, the shadow side of the MCP procedure is demonstrated and an alternative resolution is proposed.

We have considered an elementary test example, in which the regular component of the classical Green's function

$$G(z,\zeta)=\frac{1}{2\pi}\ln\frac{|z\bar{\zeta}-a^2|}{a|z-\zeta|}$$

for a disk of radius a, has been computed by the double-layer potential version of the MCP. The approximate solution of the integral equation in (4) was computed by reducing to a system of linear algebraic equations with the quadrature formulas method (the trapezoidal rule with various number k of uniform partitioning of Λ has been used).

Table 1 exhibits some results of this numerical experiment for a unit ($a=1$) disk, with the source point $\zeta=(\rho,\psi)$ being fixed at $\rho=0.3$ and $\psi=0$. The values of $G(z,\zeta)$ have been computed at several observation (field) points that are located on the radius $\varphi=0$ of the disk.

Table 1. Values of $G(z,\zeta)$ computed by the double-layer potential

Field	Partitioning number, k					Exact
point, r	10	20	50	100	200	values
0.00	.19162	.19162	.19162	.19162	.19162	.19162
0.40	.34616	.34612	.34612	.34612	.34612	.34612
0.80	.05243	.06526	.06664	.06664	.06664	.06664
0.95	-.15462	-.04842	.00568	.01449	.01502	.01517
0.99	-1.0709	-.50703	-.17094	-.06258	-.01459	.00297

From the exhibited data, it is evident that the accuracy level of the computed values of $G(z,\zeta)$ varies within the region. For any value of k, there exists the near-boundary zone, within which the computed values of $G(z,\zeta)$ notably diverge from the exact ones. Although the zone shrinks for greater k's, it still remains relatively large ($r>0.95$), even for $k=200$.

The effect of accuracy loss at the observation point approaching the boundary (we call it the *near-boundary phenomenon*) is not associated with the low accuracy level of the trapezoidal rule that was used for numerical solution of (4). The use of more accurate quadrature formulas does not radically improve the situation, because the actual cause of the phenomenon is directly related to the jump-property of the double-layer potential. Hence, the accuracy loss occurs not when eqn (4) is solved but rather when the potential (3) is computed.

1.2 Single-layer-type potential approach

Since the classical double-layer potential method is not accurate enough in computing Green's functions, an alternative approach is suggested. We picked up another potential method rooted in the *method of functional equations* [4] that in recent years is quite frequently referred to as the *method of fundamental solutions* [9]. That is, the regular component $g(z,\zeta^*)$ of a Green's function is sought as the single-layer-type potential

$$g(z,\zeta^*)=\int_{\Lambda_0}\ln\frac{1}{|z-t|}\,q(t)d_t\Lambda_0,\qquad z\in\Omega,\ t\in\Lambda_0 \tag{5}$$

where Λ_0 is a fictitious contour representing a smooth closed curve containing the actual contour Λ of Ω in its interior.

Since $g(z,\zeta^*)$ as given by eqn (5), is harmonic everywhere in Ω, the following functional equation

$$\int_{\Lambda_0} \ln \frac{1}{|z-t|}\, q(t) d_t\Lambda_0 = -T(z,\zeta^*), \qquad z\in\Lambda,\ t\in\Lambda_0 \tag{6}$$

appears while satisfying the boundary condition of eqn (2).

We call this approach the *single-layer-type potential version* of the MCP. Our experience reveals that, for a wide range of shapes, the best regularizing effect is attained when Λ_0 is just a circle of radius of the order of 0.6 to 1.0 of the greatest diameter of Ω, with the distance from Λ_0 to the closest point of the actual contour Λ being greater than at least 0.05 to 0.10 of the greatest diameter of Ω.

To demonstrate the accuracy level of this approach, the values of $G(z,\zeta)$ for a unit disk have again been computed at the same set of field points as earlier. Shown in Table 2 are the results of this experiment.

Table 2. Values of $G(z,\zeta)$ obtained by the single-layer-type potential

Field point, r	Partitioning number, k					Exact values
	10	20	50	100	200	
0.00	.19162	.19162	.19162	.19162	.19162	.19162
0.40	.34614	.34612	.34612	.34612	.34612	.34612
0.80	.06668	.06664	.06664	.06664	.06664	.06664
0.95	.01519	.01517	.01517	.01517	.01517	.01517
0.99	.00298	.00297	.00297	.00297	.00297	.00297

Two essential observations follow from the data of Table 2. First, no near-boundary phenomenon has been recorded and the accuracy level attained here is uniform for the entire region. Second, this approach is more accurate overall.

1.3 The MMP approach

In many settings, an alternative approach can be suggested for computing Green's functions. To introduce the approach (referred to herein as the *method of modified potential* (MMP)), we turn to the problem of finding the Green's function $G(z,\zeta)$ of Dirichlet problem posed over a double-connected region Ω whose outer contour is $\Lambda \subset C$, while the inner contour is $\Gamma \subset C^{(1)}$.

Let $G_0(z,\zeta)$ represent the Green's function of Dirichlet problem over the simply-connected region Ω^* bounded with Λ. As suggested in [10], we decompose $G(z,\zeta)$ onto the regular $g(z,\zeta)$ and the singular component as

$$G(z,\zeta) = G_0(z,\zeta) + g(z,\zeta).$$

With this, $g(z,\zeta^*)$, is harmonic in Ω for any position ζ^* of the source point, and, since $G_0(z,\zeta^*)$ vanishes on Λ, $g(z,\zeta^*)$ satisfies the boundary conditions

$$g(z,\zeta^*)=0, \quad z\in\Lambda \tag{7}$$

$$g(z,\zeta^*)=-G_0(z,\zeta^*), \quad z\in\Gamma \tag{8}$$

The solution to the problem (1), (7), and (8) can be written in a form of the modified single-layer-type potential

$$g(z,\zeta^*)=\int_{\Gamma_0} G_0(z,t)q(t)d_t\Gamma_0, \quad z\in\Omega \tag{9}$$

where the fictitious contour Γ_0 is embraced with Γ.

The above potential vanishes on Λ because of the defining property of $G_0(z,\zeta)$. The density $q(t)$ can be determined by taking the observation point z to the actual inner contour Γ yielding the following functional equation

$$\int_{\Gamma_0} G_0(z,t)q(t)d_t\Gamma_0=-G_0(z,\zeta^*), \quad z\in\Gamma$$

To our best knowledge, there is no a single boundary value problem for Laplace equation formulated on a multi-connected region, whose Green's function is suitable for immediate computations.

We have conducted a computational experiment to check out the effectiveness of the MMP procedure. We used it along with the single-layer-type MCP procedure to compute values of the Green's function of the Dirichlet problem for a double-connected region whose outer contour is a unit circle centered at the origin while the inner contour is a circle of radius 0.4 centered at $(-0.4, 0.0)$. Exhibited in Table 3 are some data that reveal weak and strong points of both the approaches and show their comparative degree of productivity. The values of $G(z,\zeta^*)$ are shown as computed at a set of the field points uniformly spaced on the segment $x=0$, $0\le y\le 1$, with the source point ζ^* being fixed at $(0.0, 0.75)$. The fictitious contour Γ_0 was chosen as a circle of radius 0.3 concentric with Γ.

Table 3. Comparison of the MCP and MMP output

Field point,	MCP			MMP		
	Partitioning number, k					
y	10	20	50	10	20	50
0.0	-.000280	-.000128	-.000067	-.000107	.000049	-.000032
0.2	.010667	.010712	.010781	.010700	.010779	.010798
0.4	.062359	.062411	.062443	.062407	.062435	.062448
0.6	.177534	.177574	.177585	.177583	.177590	.177593
0.8	.317893	.317902	.317911	.317907	.317913	.317914
1.0	.000014	.000006	.000002	.000000	.000000	.000000

It is worth noting that to adequately interprete the data in Table 3, it is important to realize that the same partitioning parameter k yields different computational expenses for the MCP and MMP procedures. Indeed, the dimension of the resultant system of linear algebraic equations is, in the first case, $2k \times 2k$, whereas in the second case it is $k \times k$.

2 Biharmonic Equation

The emphasis here is on boundary value problems for the biharmonic equation simulating the bending of a thin plate having uniform thickness and made of an isotropic homogeneous material. Let the plate's middle surface occupy a simply-connected region Ω, and the plate's smooth contour Λ be clamped. The bending of such a plate can be (Timoshenko and Woinowsky-Krieger [2]) described with the problem

$$\Delta\Delta u(z) = 0, \quad z \in \Omega \tag{10}$$

$$u(z) = 0, \quad \frac{\partial u(z)}{\partial n} = 0, \quad z \in \Lambda \tag{11}$$

where Δ is the Laplace operator and n represents the normal to Λ at z.

2.1 Singe-layer-type potential procedure

Let the source point in $G(z,\zeta)$ of the problem (10) and (11) be arbitrarily fixed at $\zeta^* \in \Omega$. To compute values of $G(z,\zeta^*)$, the latter is decomposed as

$$G(z,\zeta^*) = \frac{1}{8\pi}|z-\zeta^*|^2 \ln|z-\zeta^*| + g(z,\zeta^*) \tag{12}$$

The regular component $g(z,\zeta^*)$ must clearly be the solution to

$$\Delta\Delta g(z,\zeta^*) = 0, \quad z \in \Omega$$

$$g(z,\zeta^*) = -\frac{1}{8\pi}|z-\zeta^*|^2 \ln|z-\zeta^*|, \quad z \in \Lambda \tag{13}$$

$$\frac{\partial}{\partial n} g(z,\zeta^*) = -\frac{1}{8\pi}\frac{\partial}{\partial n}\left(|z-\zeta^*|^2 \ln|z-\zeta^*|\right), \quad z \in \Lambda \tag{14}$$

We look for $g(z,\zeta^*)$ in a form of the single-layer-type potential

$$g(z,\zeta^*) = \int_{\Lambda_0}\left[|z-t|^2 \ln|z-t| q_1(t) + \frac{\partial}{\partial \nu}\left(|z-t|^2 \ln|z-t|\right) q_2(t)\right] d_t\Lambda_0, \quad z \in \Omega \tag{15}$$

where Λ_0 represents a fictitious contour that embodies Λ and ν denotes the normal to Λ_0 at t. The densities $q_1(t)$ and $q_2(t)$ can be determined when

satisfying the conditions in eqns (13) and (14). In doing so, one arrives at the system

$$\int_{\Lambda_0} \begin{pmatrix} K_{11}(z,t) & K_{12}(z,t) \\ K_{21}(z,t) & K_{22}(z,t) \end{pmatrix} \begin{pmatrix} q_1(t) \\ q_2(t) \end{pmatrix} d_t\Lambda_0 = \begin{pmatrix} f_1(z) \\ f_2(z) \end{pmatrix}, \quad z \in \Lambda \tag{16}$$

of functional (integral) equations in $q_1(t)$ and $q_2(t)$. Here we have

$$K_{11}(z,t) = |z-t|^2 \ln|z-t|, \qquad K_{12}(z,t) = \frac{\partial}{\partial \nu} K_{11}(z,t),$$

$$K_{21}(z,t) = \frac{\partial}{\partial n} K_{11}(z,t), \qquad K_{22}(z,t) = \frac{\partial^2}{\partial n \partial \nu} K_{11}(z,t),$$

and

$$f_1(z) = -\frac{1}{8\pi}|z-\zeta^*|^2 \ln|z-\zeta^*|, \qquad f_2(z) = \frac{\partial}{\partial n} f_1(z),$$

with n being the normal at z to the actual contour Λ.

System (16) can be numerically solved by the quadrature formulas method (the trapezoidal rule with uniform partitioning of Λ and Λ_0 has been used in our later examples).

This approach is readily extendable to other types of edge conditions imposed on Λ. We apply it here, for instance, to a simply supported plate.

To illustrate the effectiveness of the approach, two validation examples have been considered for a circular plate. Approximate values were computed of the influence functions $G_{cl}(z,\zeta)$ (clamped edge) and $G_{ss}(z,\zeta)$ (simply supported edge). Those were tested with the aid of the analytic expressions available for $G_{cl}(z,\zeta)$ in Lebedev, Skal'skaya, and Uflyand [5] and for $G_{ss}(z,\zeta)$ in [10].

Table 4. $G_{cl}(z,\zeta)$ and $G_{ss}(z,\zeta)$ obtained by the single-layer-type potential

Field point,	$G_{cl}(z,\zeta)$			$G_{ss}(z,\zeta)$		
	Partitioning number, k					
x	10	50	exact	10	50	exact
-1.0	.000000	.000000	.000000	.000000	.000000	.000000
-0.8	.000752	.000750	.000750	.009115	.009122	.009122
-0.6	.002860	.002856	.002856	.018392	.018404	.018405
-0.4	.006046	.006044	.006044	.027371	.027379	.027380
-0.2	.009903	.009905	.009905	.035420	.035424	.035424
0.0	.013784	.013792	.013792	.041640	.041645	.041645
0.2	.016464	.016488	.016488	.044532	.044539	.044539
0.4	.014316	.014342	.014342	.040128	.040131	.040131
0.6	.007960	.007986	.007986	.028658	.028665	.028665
0.8	.002340	.002352	.002352	.014612	.014619	.014619
1.0	.000000	.000000	.000000	.000000	.000000	.000000

The computation has been carried out for a plate of unit radius on a set of the field points uniformly spaced on the diameter ($-1 \le x \le 1$). The source point ζ was fixed, in both cases, at $\zeta^* = (0.3, 0.0)$. Shown in Table 4 are the results of this experiment. The exhibited data demonstrate rather high degree of accuracy and convergence rate of the method.

Notice also that partial derivatives of the potential in eqn (15) are analytically computable. Hence, the stress components, generated by a point force in the plate, can be computed with the same accuracy level as that attained for the deflection function.

2.2 The MMP procedure

The version of the MMP procedure developed in Section 1.3, is extended here to plate problems. Consider the boundary value problem

$$B_{\lambda_1} u(z) = 0, \quad B_{\lambda_2} u(z) = 0, \quad z \in \Lambda \tag{17}$$

$$B_{\gamma_1} u(z) = 0, \quad B_{\gamma_2} u(z) = 0, \quad z \in \Gamma \tag{18}$$

stated for eqn (10) on the double-connected region Ω whose outer contour $\Lambda \subset C$, while the inner contour $\Gamma \subset C^{(1)}$. We assume the well-posedness of the problem (10), (17), and (18).

Suppose that $G_0(z, \zeta)$ represents the Green's function of the problem posed with eqns (10) and (17) over the simply-connected region Ω^* bounded with Λ. Analogously to Section 1.3, we break down the Green's function $G(z, \zeta)$ of the problem in eqns (10), (17), and (18) onto the singular $G_0(z, \zeta)$ and the regular $g(z, \zeta)$ component as

$$G(z, \zeta) = G_0(z, \zeta) + g(z, \zeta).$$

Since $G_0(z, \zeta^*)$ satisfies the boundary conditions imposed with eqn (17) for any position $\zeta^*$of the source point, the regular component $g(z, \zeta^*)$, as a function of z, must be the solution to the problem

$$\Delta\Delta g(z, \zeta^*) = 0, \quad z \in \Omega$$

$$B_{\lambda_1} g(z, \zeta^*) = 0, \quad B_{\lambda_2} g(z, \zeta^*) = 0, \quad z \in \Lambda \tag{19}$$

$$B_{\gamma_1} g(z, \zeta^*) = T_{\gamma_1}(z), \quad B_{\gamma_2} g(z, \zeta^*) = T_{\gamma_2}(z), \quad z \in \Lambda \tag{20}$$

where $T_{\gamma_1}(z) = -B_{\gamma_1} G_0(z, \zeta^*)$ and $T_{\gamma_2}(z) = -B_{\gamma_2} G_0(z, \zeta^*)$.

We look for $g(z, \zeta^*)$ in a form of the following modified biharmonic single-layer-type potential

$$g(z, \zeta^*) = \int_{\Gamma_0} \left[G_0(z, t) q_1(t) + \frac{\partial G_0(z, t)}{\partial \nu} q_2(t) \right] d_t \Gamma_0, \quad z \in \Omega \tag{21}$$

The above representation satisfies the boundary conditions of eqn (19) because of the defining properties of $G_0(z,\zeta)$. To determine the densities $q_1(t)$ and $q_2(t)$, the boundary conditions of eqn (20) can be employed. This yields the system in eqn (16) for $q_1(t)$ and $q_2(t)$, where the entries of the kernel-matrix are defined in this case as

$$K_{11}(z,t)=B_{\gamma_1}G_0(z,t), \qquad K_{12}(z,t)=\frac{\partial}{\partial\nu}B_{\gamma_1}G_0(z,t),$$

$$K_{21}(z,t)=B_{\gamma_2}G_0(z,t), \qquad K_{12}(z,t)=\frac{\partial}{\partial\nu}B_{\gamma_2}G_0(z,t),$$

and the components of the right-hand side vector are given as

$$f_1(z)=T_{\gamma_1}(z), \qquad f_2(z)=T_{\gamma_2}(z),$$

Once the solution of the system in eqn (16) is found, values of $G(z,\zeta^*)$ (which represent the plate's deflection due to the transverse unit force concentrated at ζ^*) along with values of its derivatives, required for obtaining stress components, can be accurately computed at any point $z\in\Omega$.

In Table 5, one finds the values of $G(z,\zeta^*)$ computed by the MMP for the plate whose middle plane occupies the ring-shaped region $\Omega=\{(r,\varphi):$ $0.4\le r\le 1.0,\ 0\le\varphi<2\pi\}$. The inner contour ($r=0.4$) is clamped while the outer contour ($r=1.0$) is simply supported. The point force is applied at $\zeta^*=(0.75,0.0)$. The Green's function $G_{ss}(z,\zeta)$ for the circular simply supported plate, whose analytic expression was derived in [10], has been used as $G_0(z,\zeta)$ in the potential of eqn (21).

The computed data have been obtained on two sets of field points. The points of the first set are uniformly spaced on the radius $\varphi=0$ from the inner contour ($r=0.4$) of the ring through its outer contour ($r=1.0$) (see the left fragment of Table 5). The points of the second set are uniformly spaced on the circle $r=0.75$ from $\varphi=0.0$ through $\varphi=\pi/3$ (the right fragment). The system in eqn (16) was solved by the trapezoidal rule. The circle of radius 0.3 centered at the origin has been chosen in this case as the fictitious contour Γ_0.

Table 5. Deflection $G(z,\zeta^*)$ of the ring-shaped plate under a point force

Point	k		
r	10	20	50
0.4	.00000	.00000	.00000
0.5	.00081	.00083	.00083
0.6	.00237	.00240	.00240
0.7	.00374	.00377	.00377
0.8	.00367	.00369	.00369
0.9	.00207	.00208	.00208
1.0	.00000	.00000	.00000

Point	k		
φ	10	20	50
0.0	.00402	.00404	.00404
$\pi/18$	.00305	.00307	.00307
$\pi/9$	.00185	.00186	.00186
$\pi/6$	.00099	.00100	.00100
$2\pi/9$	.00047	.00048	.00048
$5\pi/18$	.00018	.00021	.00021
$\pi/3$	.00006	.00007	.00008

Unfortunately, the above data cannot be directly checked out, because the exact expression for the influence function under consideration is not available. However, the above table reveals high convergence rate of the MMP algorithm (indeed, the data in the columns of $k = 20$ and $k = 50$ differ by only the last of the exhibited decimal points). Hence, given the fact that the potential in eqn (21) represents a biharmonic function, which exactly satisfies the boundary conditions on the outer contour Λ, the indicated convergence indirectly confirms the authenticity of the computed data.

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