Boundary element analysis of buried pipe – soil interaction

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Abstract

The mixed boundary value problem for a pipe embedded in soil is solved with two-dimensional modeling. In this type of problem the external stress vector is specified on a part of the surface and the displacement on the other part that constitutes the pipe-soil interface. The materials are assumed to be elastic. It is also assumed that the problem domain is multiply connected. Numerical results are presented for typical examples.

Introduction

Underground pipelines should support the soil loading and the loads applied on the ground surface. The intensity of surface loads decreases with increasing depth. So the effect of traffic or other surface loads on buried pipelines is relatively minor if pipelines are buried deeply. But these loads have significant effects on shallow buried pipelines.

The weight of the pipeline and the fluid transported do not affect self to the overall stress in the pipeline in a significant manner. So, these loads are neglected in this study.

Formulation of the boundary element method for soil-structure interaction

The domain is combined of two regions as shown in Figure 1: compacted sand
backfill and the pipe. The objective is to determine the displacement vectors and the stress tensors in both the regions and on the boundary.

![Figure 1. Model of the pipe-soil system](image)

Let $V$ be a region filled by a linear elastic material and $S$ its boundary. In plane problems, $S$ and $V$ are the closed curves and a plane region, respectively.

In the standard formulation of the boundary element method, any $i$th component of the displacement vector ($i=1,2$) at a point $y$ in the region $V$ can be expressed as follows:

$$
\begin{align*}
    u_i(y) &= \int t(x) u^i(x, y) ds - \int t^i(x, y) u(x) ds \\
    &+ \int t(x) u^i(x, y) rd\theta - \int t^i(x, y) u(x) rd\theta \\
    &+ \int t^i(x, y) u(x) ds \\
    &- \int t(x) u^i(x, y) ds
\end{align*}
$$

(1)

where, $t(x)$ is the surface traction vector on the boundary $S$, $r$ the radius. The kernels $u^i(x, y)$ and $t^i(x, y)$ are given below,

$$
\begin{align*}
    u^i_{j}(x, y) &= -\frac{1}{8\pi G(1-\nu)} \left\{(3-4\nu) \ln r d_{ij} - \frac{(x_i - y_i)(x_j - y_j)}{r^2} \right\}
\end{align*}
$$

(2)
\[ T^i_{jk} (x, y) = -\frac{1}{4\pi(1-\nu)} \left\{ (1-2\nu)\left( \frac{(x_i-y_i)}{r^2} \delta_{jk} + \frac{(x_k-y_k)}{r^2} \delta_{ij} \right) \right. \\
+ \left. \frac{(x_j-y_j)}{r^2} \delta_{ik} \right\} \left( \frac{2(x_i-y_i)(x_j-y_j)(x_k-y_k)}{r^4} \right) \] 

\[ (i, j, k = 1,2) \]

\[ t^i_{\sim} (x, y) = T^i_{\sim} (x, y) n(x) \]

\[ r^2 = (x_i-y_i)(x_i-y_i) \]

where \( G \) and \( \nu \) represent the shear modulus and Poisson's ratio, respectively. \( n(x) \) is the unit outward normal vector of the surface \( S \) at the point \( (x) \). \( \delta_{ij} (i, j = 1,2) \) is the Kronecker delta \([1,2,3]\). The summation convention has been used in all the expressions.

For a multiply connected region, the boundary \( S \) contains a finite number of disjoint curves, and the integral over \( S \) is reduced to the summation of the integrals over this disjoint curves. From equation (1), it is clear that to obtain the displacement vector on the boundary it is enough to determine the unknown displacement components and the tractions at any point \( y \) of \( V \).

For solving the integral equation (1), the soil is assumed to be elastic, and the boundary \( S \) is idealized as an ensemble of line segments on the external boundary of the soil and circular segments on the interior of the pipe and the pipe-soil interface. If the number of segments are assumed to be \( N \) on each boundary, the number of the end points will also be \( N \) on each boundary.

It is assumed that the variation of the displacement components and the tractions on any of these segments is linear. Then the unknowns of the problem are reduced to the values of the displacement components or the tractions on the nodal points.

In this problem, there is a uniform vertical load at the top of the sand. There is a linear varying horizontal loading on the vertical edges of the finite sand domain, and no vertical loading. It is assumed that there is no horizontal movement at the vertical centerline \([4]\) and the bottom is rigid. Both displacement components and traction components are unknown on the pipe’s external boundary. The displacement components on the internal boundary of the pipe are also unknown. The forces inside the pipeline are neglected. For \( N \) nodal points, \( 2N \) integral equations can be written by assuming there is a singular loading at every nodal point in each direction on each boundary. Therefore, for four boundaries, one external, two at the interface, and one internal, \( 8N \) equations are obtained. In these integral equations, the integrals over the boundary are reduced to the summation of the integrals over the segments \([5,6]\).
A new artificial boundary, which includes line segments but not the nodal points, is defined. Around each nodal point, a small arbitrary curvilinear part which leaves the nodal point outside, is added to complete this new artificial boundary as shown in Figure 2. It is assumed that displacement components are constant, but no stress on this small curvilinear parts.

![Figure 2. New artificial boundary](image)

After necessary calculations, these small curvilinear parts are shrunk to the nodal points. Calculating integrals over this artificial boundary gives a linear system of $8N$ equations with $8N$ unknowns which are the displacement components or the tractions at nodal points.

The tractions are equal, but opposite in sign on the boundary between the soil and the pipeline, and the displacement components are of the same sign and magnitude. In view of symmetry about the vertical centerline, only half of the equations need to be solved.

After solving this linear system using the displacement field and the artificial boundary by the help of constitutive equations[7,8], the stress components can be calculated at any arbitrary internal point $y$ for segments as follows:

$$ T_{ij}(y) = \sum_{J=1}^{N} \left[ \frac{u_k(J+1) - u_k(J)}{\theta_{J+1} - \theta_J} \right] \left[ \frac{\theta_{J+1}}{\theta_J} w_{ij}^k(x, y) d\theta + \frac{\theta_{J+1}}{\theta_J} u_{ij}^k(x, y) r_k(x) d\theta \right] \tag{5} $$
The expressions of the kernels in eq. (5) are given below.

\[
\begin{align*}
\mathbf{u}_{ij}^k (x, y) &= -\frac{1}{4\pi(1-\nu)} \left\{ (1-2\nu)(-\frac{x_i-y_i}{r^2}\delta_{jk} + \frac{x_k-y_k}{r^2}\delta_{ij}) \right. \\
&\quad - \frac{x_j-y_j}{r^2}\delta_{ik} - \frac{2(x_i-y_i)(x_j-y_j)(x_k-y_k)}{r^4} \left. \right\} \\
\mathbf{w}_{ij}^k (x, y) &= \frac{G}{4\pi(1-\nu)} \left\{ -4\nu \varepsilon_{km} \frac{x_m-y_m}{r^2}\delta_{ij} + 2\varepsilon_{km} \frac{x_m-y_m}{r^2}\delta_{ij} + \\
&\quad 2(1-\nu)\varepsilon_{jm}\frac{x_m-y_m}{r^2}\delta_{ik} - \varepsilon_{jm} \frac{x_m-y_m}{r^2}\delta_{jk} \right. \\
&\quad \varepsilon_{ik}\frac{x_j-y_j}{r^2} + \varepsilon_{jk}\frac{x_i-y_i}{r^2} - 4\varepsilon_{km} \frac{x_m-y_m}{r^4} \left. \right\}
\end{align*}
\] (6)

where \(\varepsilon\) is the permutation symbol.

The stress component \(T_{\theta\theta}(y)\) in the \(r, \theta\) system at any internal point \(y\) can be calculated as follows:

\[
T_{\theta\theta}(y) = T_{11}(y)n_2^2(J) + T_{22}(y)n_1^2(J) - 2T_{12}(y)n_1(J)n_2(J)
\] (7)

where \(n_1(J)\) and \(n_2(J)\) denote the components of \(n(J)\), the unit outward normals of a segment \(J\). Although the unit outward normal is constant on the linear segments it varies along the points on the circular segments. Eq.(8) can be written as follows:

\[
T_{\theta\theta}(y) = \int_S [u_{k\theta\theta}^\theta (x, y) \mathbf{t}_k(x) \mathbf{R} + w_{k\theta\theta}^\theta (x, y) \frac{\partial}{\partial \theta} \mathbf{u}_k(x, y)] R \theta
\] (8)

where the kernels of \(u_{k\theta\theta}^\theta (x, y)\) and \(w_{k\theta\theta}^\theta (x, y)\) are as follows:

\[
\begin{align*}
u_{k\theta\theta}^\theta (x, y) &= u_{k\theta\theta}^{11}(x, y)n_2^2(J) + u_{k\theta\theta}^{22}(x, y)n_1^2(J) - 2u_{k\theta\theta}^{12}(x, y)n_1(J)n_2(J) \quad (10) \\
w_{k\theta\theta}^\theta (x, y) &= w_{k\theta\theta}^{11}(x, y)n_2^2(J) + w_{k\theta\theta}^{22}(x, y)n_1^2(J) - 2w_{k\theta\theta}^{12}(x, y)n_1(J)n_2(J) \quad (11)
\end{align*}
\]
Stress components on the boundary have $1/r$ singularity. To eliminate this singularity a new coordinate system is defined as shown in Figure 3. An interior point $C$ is considered. The nearest point of the boundary to $C$ is a point $B$ on the circular segment $J$. $B$ lies between the $\theta_{Jth}$ and $\theta_{J+1th}$ angles for the $Jth$ and $(J+1)th$ nodal points. But, this introduces a restriction that the point $B$ should be neither the $Jth$ or the $(J+1)th$ nodal point. At this point $C$, the stress component $T_{\theta\theta}(y)$ will be calculated and $\varepsilon$ set to zero in the limit. The stress component $T_{\theta\theta}(y)$ is now calculated without any singularity at a point on the $Jth$ circular segment of the artificial boundary.

There is an angle $\theta_0$ between this specific point and $\theta_J$. $\theta_0$ can be equal to neither $\theta_J$ nor $\theta_{J+1}$.

**Numerical Examples**

The selected values for sample problems are given in Table 1.

<table>
<thead>
<tr>
<th>$r_1$ (ft)</th>
<th>$r_2$ (ft)</th>
<th>$L$ (ft)</th>
<th>$h$ (ft)</th>
<th>$\nu$ (pipe)</th>
<th>$\nu$ (sand)</th>
<th>$E$ (pipe) (ksi)</th>
<th>$E$ (sand) (psi)</th>
<th>$\gamma$ (lb/ft$^3$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.25</td>
<td>0.30</td>
<td>0.85</td>
<td>0.85</td>
<td>0.4</td>
<td>0.25</td>
<td>28000</td>
<td>13750</td>
<td>108</td>
</tr>
</tbody>
</table>
where \( r_1 \) and \( r_2 \) are the internal and external radii of the pipeline, respectively and \( E \) and \( \gamma \) are the modulus of elasticity and the specific weight of the soil, respectively. The surface loads, \( \sigma_\theta \), are chosen as 0, 45, and 90 (psi) for Case 1, Case 2, and Case 3, respectively.

The results are given between the crown and the invert in Figures 4 and 5 for the displacement components.

**Figure 4. The variation of \( u_1 \) (in)**

**Figure 5. The variation of \( u_2 \) (in)**
The results are given between the crown and the invert in Figure 6, 7 and 8 for the stress components.

Figure 6. The variation of $T_{11}$ (psi)

Figure 7. The variation of $T_{12}$ (psi)
The variation of the displacement components at the top of the sand are given in Figure 9 and 10.

**Figure 8. Variation of T_{22} (psi)**

**Figure 9. Variation of u_1**

**Figure 10. Variation of u_2**
The variation of the stress components on the vertical centerline between the crown and surface of the sand are given in Figures 11 and 12.

**Conclusions**

It is known that the magnitude of the measured stress in a buried pipe with uniform loading on the surface is maximum in the shoulder region, i.e. the wall between the crown and the spring line, [4,9]. The results of this study indicate the same trend. The stress and deflection values are symmetrical as expected with an error of less than 1%. This error may be decreased by increasing the number of nodal points.

In this study, a program was developed to obtain the displacements and the stress components by boundary element plane strain analysis of a buried conduit. The results were obtained for 64 nodal points. The program can be used for plane stress problems by modifying the Poisson’s ratio. The program needs to be compiled for the dimensions, shear modulus, Poisson’s ratios, and the external loading.
Acknowledgement

The authors would like to thank Florida Department of Transportation (Contract Monitor: Mr. R. G. Powers, Materials Division) for partial financial support. Thanks are due to Dr. S.E. Dunn, Prof. and Chairman, Department of Ocean Engineering and Dr. J.S. Jurewicz, Dean of Engineering, Florida Atlantic University for their support and encouragement.

References


