A GEM-FEM study of fluid structure interaction

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Abstract
A Green element method (GEM)-Finite element method (FEM) combination (GEM-FEM) is applied to calculate hydrodynamic pressures arising from fluid-structure interaction in a reservoir impounded by a dam. The radiation condition for an infinite reservoir is efficiently handled by GEM. Some examples demonstrating the efficiency and utility of this GEM-FEM numerical combination are presented.

1 Introduction
The interaction of gravity dams with water is highly influenced by the hydrodynamic pressures generated on the upstream face. The study of this effect will facilitate a general understanding of the pressures, displacement and stresses that accompany a fluid-structure interaction between a dam and an impounding reservoir. In addition the results so obtained will provide a quantitative information concerning the response of a structure to different modes of excitation.

A significant challenge in this work, is the accurate representation of the so called far field boundary condition. It has always been routine, to represent the infinite boundary in the fluid domain by an artificial boundary placed at a finite distance away from the structure. As has been observed with the finite element method, this has resulted in inaccurate solutions. It is therefore proper, to seek means of eliminating the wave reflections that lead to spurious numerical results by taking a proper account of the Sommerfeld radiation condition. To deal with this problem, we have opted to employ the Green element method (GEM) to numerically solve the fluid domain. We shall demonstrate that GEM’s hybrid boundary element-finite element combination makes it ideal to handle problems of this nature. While its boundary element formulation facilitates its ability to handle the fluid’s far boundary, its finite element discretization approach simplifies its
coupling with the finite element at the structure interface. Unlike other so called hybrid methods, there is no need to establish special compatibility conditions at the common boundary, neither is there any need to tamper with the boundary conditions, instead GEM's hybrid formulation guarantees that a one-to-one relationship is a priorily established between the two methods at the common boundary. We shall now go ahead and establish the governing equations for the fluid-structure system, and carry out their discretizations with GEM and FEM numerical techniques.

2 The governing equations and their discretization

For the fluid domain, we consider an inviscid, irrotational incompressible fluid governed by the Laplace equation:

$$\nabla^2 \phi = 0$$  

(1)

where $\phi$ is the hydrodynamic pressure distribution (in excess of the hydrostatic pressure). The hydrodynamic pressure for a 2-D fluid can be obtained by solving equation(1) subject to the following boundary conditions: (i) For negligible surface waves, the free surface is governed by:

$$\phi = 0$$  

(2a)

At the fluid-structure interface:

$$\frac{\partial \phi}{\partial n} = -\rho a_n$$  

(2b)

where $\rho$ is the fluid density, $a_n$ is the acceleration imparted on the structure in a direction outwardly-normal to the solid boundary. The boundary condition at the floor of the reservoir is:

$$\frac{\partial \phi}{\partial y} = 0$$  

(2c)

In order to keep faith with the Sommerfeld radiation boundary, the following boundary condition is imposed at the far field:

$$\frac{\partial \phi}{\partial n} = 0$$  

(2d)

The discretized GEM formulation of equation(1) can be written in matrix form as:

$$\sum_{e=1}^{M} (R_{ey} \Phi + L_{ey}(q/k)) = 0$$  

(3)

where $e$ is an element counter, $M$ is the total number of elements, $k$ is the
conductivity (it takes a value of unity for this problem), q is the flux of the dependent variable, $L_{ij}$ and $R_{ij}$ are matrix variables and are represented by:

$$
R_{ij}^{(e)} = \int_{\Gamma} \frac{\partial G(r,r_i)}{\partial n_{ij}} \delta_{ij} ds
$$

$$
L_{ij}^{(e)} = -\int_{\Gamma} G(r,r_i) \delta_{ij} ds
$$

(4)

where $\Omega_j$ is an interpolating function with respect to node $j$ of an element.

The equation of motion of a structure subjected to external forces can be represented in standard finite element form as:

$$
M \ddot{U} + C \dot{U} + K U = F_e + F_h
$$

(5)

where $\ddot{U}$, $\dot{U}$, $U$ are vectors of nodal acceleration, velocity and displacement respectively, $F_e$ time dependent external force on structure, and $F_h$ is hydrodynamic force resulting from fluid-structure interaction. The elemental mass matrix is given by:

$$
M^{(e)} = \int_N^T \rho_s \, N d\zeta
$$

(6)

and the structural damping matrix is:

$$
C = aM + bK
$$

(7)

where $a$, and $b$ are arbitrary constants. Employing plain strain formulation, the elementary stiffness matrix is given by:

$$
K^{e} = \sum_{\zeta} \int_B^{T} D \, B \, d\zeta
$$

(8)

where $D$ is the constitutive matrix, and $B$ is the strain-displacement matrix, and $\rho_s$ is the density of the structural material. An interactive scheme has been developed to solve the discretized equations. Field variables are transferred from the fluid to the structure at each time step. The pressure variables on the structure obtained from the fluid domain by the GEM scheme are converted into forces, and structural displacements, stresses, and accelerations are calculated by FEM. As a result of these accelerations, the boundary conditions on the structural element change. Hence at each time step, divergence free pressures from the fluid, as well as forces, displacements, stresses and accelerations are computed.

It should be noted that because of GEM's hybrid formulation, the resulting matrices obtained from both fluid and structure domains are both symmetric, and not fully populated. It is therefore relatively straightforward to handle the coefficient matrices, and at the same time consider as many nodes as we deem fit.
3 Numerical tests

We check the validity of the formulation developed herein by investigating the following numerical examples.

3.1 example 1

In this example, we aim to obtain an appropriate truncation boundary that will yield a value of pressure coefficient that is as close as possible to the analytic solution of Westergaard\(^1\). Different locations away from the vertical upstream face of an infinite reservoir were tested. For each of these locations, the 2-D problem domain was discretized, and a maximum pressure coefficient, \(C_0 = \frac{\phi_0}{\rho a_h}\) at the bottom of the structure computed. Sharan\(^2\) had earlier proposed a boundary condition for the truncation boundary that gave accurate results for distances not very far from the upstream face of the structure. Here we adopt equation (2d) which is valid for a stationary boundary to check how far away from the structure the boundary must have to be located in order to satisfy the effect of the far field. An accurate transmitting boundary is yet difficult to find without resorting to imposing the analytic solution of governing equation at the transmitting boundary (see Sharan\(^2\)). This may pose some difficulties for transient problems.

The following problem parameters are assumed.

1. Height of wall = 10.0m
2. Thickness of wall = 1.0m
3. Modulus of elasticity = \(2.0 \times 10^{10}\) N/m\(^2\)
4. Poisson’s ratio = 0.25
5. Height of fluid = 10.0m
6. Mass density of fluid = 1000 kg/m\(^3\)
7. Mass density of structure = 8750 kg/m\(^3\)

The boundary conditions are as specified in equations (2a-2d). The structure is assumed rigid, so the ensuing results are obtained from GEM computation. By adopting this approach, we wish to exploit the ability of the boundary integral technique to handle infinite domains. We start by truncating the far field at artificial boundaries some distances\(l\) away from the source of radiation, before imposing the boundary condition. Although GEM can yield reliable estimates of hydrodynamic pressures, it is expected that the accuracy will be sensitive to the length of the truncated boundary, and acceptable results restricted within a particular range of \(l/h\) values. This is where the challenge lies in dealing with practical problems involving a far boundary. For example accurate representation of a reservoir-dam system plays a significant role in the accuracy of the results obtained when determining the pressures and the stresses needed in design.
Table 1: Maximum pressure coefficients at different locations from structure

<table>
<thead>
<tr>
<th>l/h</th>
<th>Mesh Size</th>
<th>Pressure Coefficient $C_0$</th>
<th>% Error</th>
</tr>
</thead>
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<tr>
<td></td>
<td>present</td>
<td>ref. 1</td>
<td></td>
</tr>
<tr>
<td>0.5</td>
<td>20x10</td>
<td>1.181</td>
<td>0.7425</td>
</tr>
<tr>
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<td>10x10</td>
<td>0.7738</td>
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</tr>
<tr>
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<td>10x12</td>
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<td>0.7425</td>
</tr>
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<td>0.7397</td>
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<tr>
<td>2.4</td>
<td>24x10</td>
<td>0.7308</td>
<td>0.7425</td>
</tr>
<tr>
<td>2.5</td>
<td>25x10</td>
<td>0.7321</td>
<td>0.7425</td>
</tr>
</tbody>
</table>

Fig. 1 Percentage errors for truncated boundaries
Table 1 and Fig. 1 illustrate the profile of percentage errors obtained by truncating the boundaries at different positions downstream of the structure. It can be seen that the least error was obtained for \( l = 1.5h \) when compared with Westergaard's classical maximum pressure coefficient of 0.7425 for an \( l/h \) of infinity. We also observe that the percentage error significantly went down when \( l/h \) attained a value of unity, and decreased monotonically until it attained a value of 1.5 after which it shows some tendency towards instability. We can therefore rightly infer that acceptable results for problems of this type involving infinite domains could be obtained where the ratio of \( l/h \) is unity or slightly more that unity. This objective can be attained without resorting to enhanced boundary conditions.

### 3.2 Example 2

In this example, the structure is considered flexible, and as such all the physical parameters listed above do apply. A vertical and rectangular structure of 10m x 1.0m dimension is considered. The structure is discretized by 22 isoparametric linear finite elements. Fluid-structure interaction is accomplished by contact between the structure's vertical surface with an incompressible fluid. The fluid region is considered infinite, and is discretized by 150 finite elements in order to accomplish the minimum error for an unbounded domain. Simulations were carried out to study the interaction between the fluid and the structure as the specified boundary condition at the fluid-structure interface changes with time.

Fig. 2 shows the hydrodynamic pressure distribution at different times.
3.3 Example 3
An initial acceleration is applied to the vertical wall and its effects are studied. Both the fluid and the structural domains retain their previous discretization. The physical dimensions of the fluid as well as the parameters for the structure are the same as those already specified. The Newmark method is used for the solution of the system of equations, and a time step $\Delta t = 0.01$ seconds has been adopted. The hydrodynamic pressure at the structural interface, is a function of both the time dependent excitation imparted on the wall and the position of the truncation boundary inside the fluid domain. A study is carried out to study the variation of the pressure coefficient at the bottom of the tank at different times. Fig. 3 illustrates the profile of the pressure coefficient at different times. This gives us an idea of how the ratio of the maximum dynamic and static pressures vary with time as a result of transient loading on the vertical wall. We can infer from Fig. 3 that the vibration of the hydrodynamic pressure decreases with time as a result of radiation damping from the far field.
3.4 Example 4

This example is almost the same as the previous with the only difference that the total hydrodynamic pressure coefficient (total hydrodynamic pressure/total hydrostatic pressure at the on the face of the dam). Fig. 4 shows the results computed at different times. A decrease in amplitude as time increases can be observed. This can also be attributed to damping and energy loss due to truncation at far field.

4. Conclusions

This paper presents our initial attempts to use GEM to solve problems involving fluid-structure interaction problems. Even though that GEM has been applied in the past to solve challenging problems, it is possible henceforth, to extend the method to a fluid-structure problem involving an infinite domain. What this paper has emphasized is that GEM domain discretization ensures that no special interface consideration is required at the fluid-structure boundary. GEM has not only the merit of yielding a banded symmetric matrix the same as FEM, in addition compatibility and boundary condition requirements at the common interface are automatically satisfied.

