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The effect of surface tension and surface friction on the free surface flow in a two-dimensional channel

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Abstract

A boundary integral technique is developed for steady two-dimensional, irrotational, incompressible fluid flow with a free surface in a two-dimensional channel. The effects of surface tension and surface friction on the free surface profile are investigated. The numerical solutions indicate that the surface tension tends to increase the curvature of the free surface whilst the surface friction tends to smoothen out the free surface profile and it dampens the effect of the surface tension.

1 Introduction

The free surface flow of an incompressible, irrotational and inviscid fluid in a two-dimensional channel has been the subject of considerable research in engineering over many years and this is because such flows frequently occur in practical engineering problems. The difficulty which arises in all free surface flows is that the boundary of the fluid field is not known *a priori*. The problem of the free surface flow becomes even more difficult to solve mathematically if the physical effects such as gravity, surface tension or surface friction are included since the fluid velocity on the free surface depends on the profile of the free surface which is unknow before the complete solution of the problem has been found. This is because the determination of the free surface profile is part of the solution procedure and the fluid velocity on the free surface is related to the free surface profile through either a nonlinear equation or a differential equation.

For free surface flows under the influence of gravity there has been a considerable number of investigations performed using a variety of methods. Watters and Street [1] studied this problem using complex variable theory whereas the finite element method was applied to this problem by Varoglu and Finn [2] and Betts [3]. Further, various boundary integral techniques have been developed in order to obtain the solution for the free surface flow in a two-dimensional channel by Belward and Forbes [4] and Wen, Ingham and Widodo [5]. In all these investigations the free surface profile was determined for a given geometry of the bed of the channel.

A recent development of free surface flows in a two-dimensional channel is to determine the profile of the channel bed under the condition of a given free surface profile. In such inverse problems Lonyangapuo, Elliott, Ingham and Wen [6] have developed the minimal height method in order to obtain the geometry of the channel bed when the geometry of the free surface is known *a priori*.

In this paper a further development of the investigation in the free surface flow in a two-dimensional channel is the inclusion of the effects of the surface tension and the surface friction. Numerical solutions have been obtained for the situations of both upward facing and backward facing steps.

2 Formulation of the boundary-integral equation

We consider a two-dimensional steady, inviscid and irrotational fluid flow over a step in a channel under the influence of gravity, surface tension and surface friction. A coordinate system $Z = X + iY$ is introduced with the X axis along $A_\infty B$ and the Y axis being measured vertically upwards and the origin of the coordinate system is at the point where the bed surface of the channel first deviates from being horizontal. Far upstream of the step the fluid flows horizontally in the positive X direction with a uniform speed U_∞ and the depth of the fluid is h. There is a step which starts at $X = 0$ and the equation of the bed of the channel is given by $Y = f(X)$ for $0 \leq X \leq L$ and the bed of the channel is horizontal for $X \geq L$. The above assumptions allows the introduction of a velocity potential Φ and a streamfunction Ψ such that the complex velocity potential $W = \Phi + \Psi$ is an analytical function in the domain which is occupied by the fluid. On the bed of the channel Γ_1, the streamfunction, Ψ, is chosen, without any loss in generality, such that $\Psi = 0$, whereas on the free surface, Γ_2, we take $\psi = q = hU_\infty$.

The problem is non-dimensionalized using by the transformations

$$x + iy = (X + iY)/h \qquad w = \psi + i\phi = (\Psi + i\Phi)/(hU_\infty) \tag{1}$$

and the governing equation is the Laplace equation, namely

$$\nabla^2 \Psi = \nabla^2 \Phi = 0 \tag{2}$$

On the bottom surface

$$\Psi = 0, \qquad \frac{\partial \Phi}{\partial n} = 0 \tag{3}$$

and on the free surface

$$\Psi = 1, \qquad \frac{\partial \Phi}{\partial n} = 0 \tag{4}$$

On the free surface we enforce the dynamic condition

$$\frac{F_r^2}{2} u^2 + y - We \frac{d\theta}{ds} + 2 \frac{F_r^2}{Re} \frac{du}{ds} = \frac{F_r^2}{2} + 1 \tag{5}$$

where $F_r = U_\infty / \sqrt{gh}$ is the Froude number, $We = T/(\rho g h^2)$ is the Weber number, $Re = \rho U_\infty h/\mu$ is the Reynolds number, ρ is the density of fluid, μ is the viscosity of the fluid, g is the magnitude of the gravitational acceleration gravity, T is the surface tension, u is the fluid speed, θ is the tangential angle that the fluid velocity vector on the free surface makes with the positive x-axis.

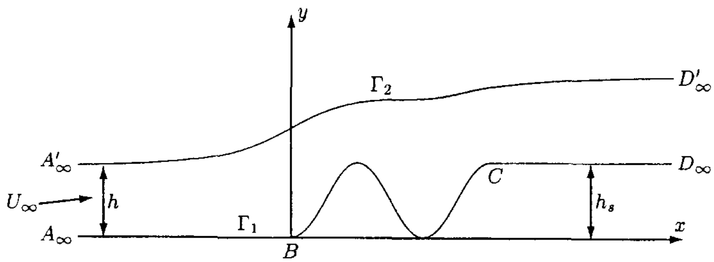

Figure 1. A schematic diagram of the geometry for the free surface fluid flow over a step.

The boundary integral equations for the fluid speed, u, on the bed of the channel and the angle, θ, on the free surface are given by (for more details see Wen, Ingham and Widodo [5]):

$$\ln u(s) = \sqrt{t(s)} \left\{ \int_{\Gamma_1} \frac{\beta(l)\eta(l)u(l)}{\sqrt{\eta(l)}[\eta(l) - t(s)]} dl \right. \\ \left. - \int_{\Gamma_2} \frac{\ln u(l)\eta(l)u(l)}{\sqrt{\eta(l)}[\eta(l) - t(s)]} dl \right\} \quad s \in \Gamma_1 \tag{6}$$

$$\theta(s) = \sqrt{t(s)}\left\{ \int_{\Gamma_1} \frac{\beta(l)\eta(l)u(l)}{\sqrt{\eta(l)}[\eta(l)-t(s)]}dl - \int_{\Gamma_2} \frac{\ln u(l)\eta(l)u(l)}{\sqrt{\eta(l)}[\eta(l)-t(s)]}dl \right\} \quad s \in \Gamma_2 \tag{7}$$

where $\beta(l)$ is the angle that the bed surface makes with the positive x-axis and $\eta(l)$ and $t(s)$ are given by

$$\eta(l) = e^{-\pi\phi(l)}, \qquad t(s) = e^{-\pi\phi(s)} \tag{8}$$

Along the free surface and the solid boundary we have $d\phi/ds = u$ and therefore the potential function may be calculated by using the equation

$$\phi(s) = \phi_A + \int_0^s u(l)dl \tag{9}$$

The coordinates of the free surface are given by $dx/ds = \cos\theta$ and $dy/ds = \sin\theta$ and therefore

$$x(s) = x'_A = + \int_0^s \cos\theta(l)dl \quad \text{and} \quad y(s) = y'_A + \int_0^s \sin\theta(l)dl \tag{10}$$

were x'_A and y'_A are the coordinates of the free surface at a point which is so far upstream that the free surface may be considered to be horizontal. When surface friction is negligible, namely $Re = \infty$, then equation (5) leads to an algebraic equation for the fluid speed on the free surface, which is given by

$$u(s) = \sqrt{1 + \frac{2}{F_r^2}(1-y) + \frac{2}{F_r^2}We\frac{d\theta}{ds}} \tag{11}$$

in which $d\theta/ds$ is calculated using a central-difference scheme.

When the surface friction is included then the fluid speed on the free surface is obtained by solving the nonlinear ordinary differential equation (5). In this paper, on the boundary we employ the finite-difference method and the central-difference scheme, which has second-order accuracy, is applied to the expression $d\theta/ds$ and a backwards finite-difference, which is characteristic of an upwind scheme for a better convergence of the iterative procedure, for the expression du/ds. Thus equation (5) becomes

$$\frac{F_r^2}{2}u_i^2 + y_i - We\frac{\theta_{i+1}-\theta_{i-1}}{2\Delta s} + 2\frac{F_r^2}{Re}\frac{u_i - u_{i-1}}{\Delta s} = \frac{F_r^2}{2} + 1 \tag{12}$$

On solving equation (12) we obtain

$$u_i = -\frac{2}{R_e\Delta s} + \frac{2}{R_e\Delta s}\sqrt{1 + \left(\frac{R_e\Delta s}{4}\right)^2 \left\{\frac{4}{R_e\Delta s}u_{i-1} + C\right\}} \tag{13}$$

where

$$C = \frac{2}{F_r^2}\left[1 + \frac{F_r^2}{2} + We\frac{\theta_{i+1} - \theta_{i-1}}{2\Delta s} - y_i\right] \tag{14}$$

A numerical iterative procedure, see Wen, Ingham and Widodo [5], has been applied to solve the system of integral equations (6), (7), (9), (10) and (11) or (13) for different situations in order to obtain the potential function on the boundaries, $\theta(s)$, the coordinates of the free surface, $x(s)$ and $y(s)$, and the fluid speed, $u(s)$, on the solid boundary and the free surface, respectively.

3 Numerical results and discussion

In order to illustrate typical results which we obtained, the numerical calculations are presented for an upward facing step whose surface is given by the expression

$$y = \frac{h_s}{2}\left[1 - \cos\left(\frac{x\pi}{l}\right)\right] \tag{15}$$

and a backward facing step whose surface is given by the expression

$$y = -\frac{h_s}{2}\left[1 - \cos\left(\frac{x\pi}{l}\right)\right] \tag{16}$$

where h_s is the step height and l is the step length. In this paper the numerical results are presented for steps with $h_s = 1.0$ and $l = 4.0$.

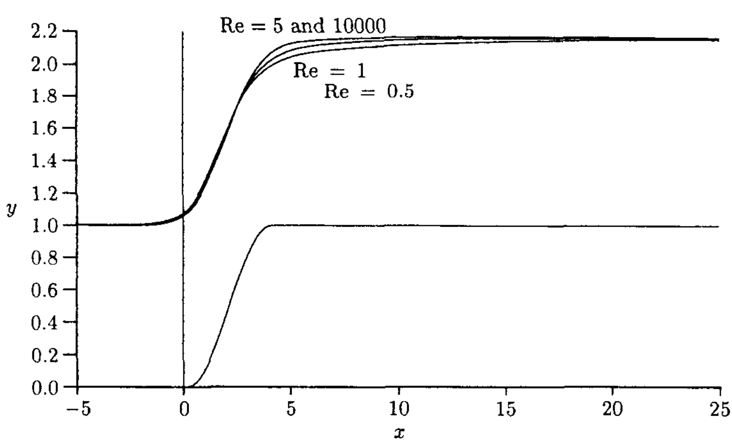

Figure 2. The free surface profile due to an upward facing step when l = 4.0, h_s = 1.0, F_r = 4.0, We = 0.0, Re = 10000, 5, 1 and 0.5.

Figure 2 shows the free surface profiles for an upward facing step subject to a flow with an upstream Froude number F_r = 4.0 and We = 0.0. As the Reynolds number decreases we can see that the fluid depth decreases as it flows over the step. Thus the effect of increasing the surface friction is to produces a more flatter free surface profile and this has occured, in particular, in the region just after step where the the fluid speed is lower than that in the region just prior to the step which allows the effect of the surface friction to become more significant.

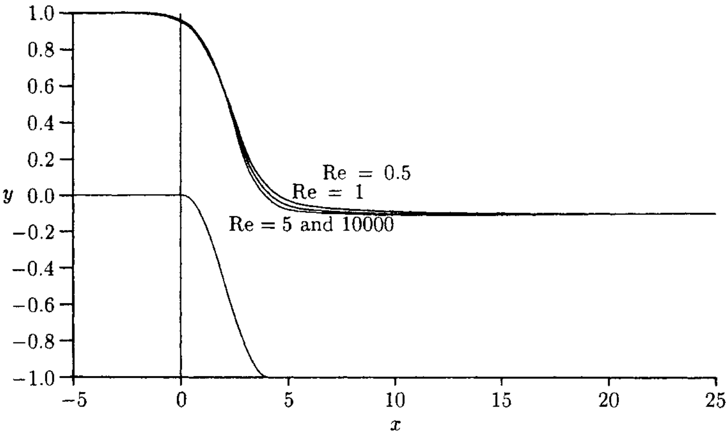

Figure 3. The free surface profile due to a backward facing step when l = 4.0, $h_s = 1.0$, $F_r = 4.0$, $We = 0.0$, Re = 10000, 5, 1 and 0.5.

Figure 3 shows the free surface profile for a fixed backward facing step which is also subject to a flow with an upstream Froude number $F_r = 4.0$ and $We = 0.0$. The flow profiles over the backward facing step shows all the characteristics of the upward facing step but to a lesser extent. The much faster fluid velocity attained from the drop in the free surface height has restricted the deviation in the free surface flow profile from that of the inviscid free surface flow profile.

What can also be seen in the Figures 2 and 3 is that for the free surface profile with Reynolds numbers $Re = 5.0$ and Re=1000 then the profiles are almost identical and it is only when the Reynolds number reaches a value as low as about $Re = 1.0$ does the free surface profile start to deviate from that obtained when $Re = \infty$ to any large degree. This shows that the surface friction only becomes significant when the Reynolds number drops below about $Re = 5.0$.

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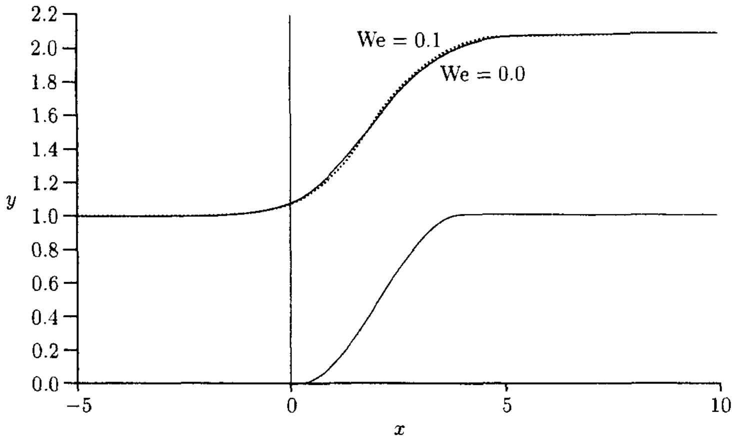

Figure 4. The free surface profile due to an upward facing step when $l = 4.0$, $h_s = 1.0$, $F_r = 4.0$ and $Re = \infty$. — $We = 0.0$, ...$We = 0.1$.

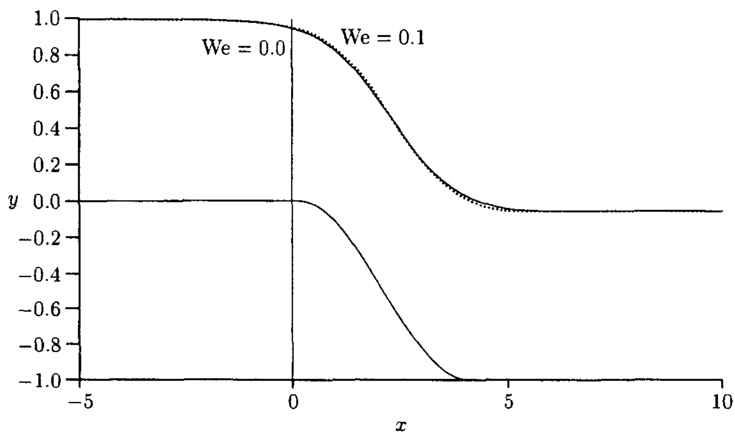

Figure 5. The free surface profile due to a backward facing step when $l = 4.0$, $h_s = 1.0$, $F_r = 4.0$ and $Re = \infty$. — $We = 0.0$, ...$We = 0.1$.

For the situation of $F_r = 4.0$, $Re = \infty$, namely no surface friction, we have investigated the effect of the surface tension on the profile of the free surface. We have found that there was no visible deviation in the free surface profile from that of the 'no surface tension' free surface profile, until the Weber number has increased up to about $We = 0.1$. Figures 4 and 5 are the free surface profiles over the upward and backward facing steps, respectively. We observe that the surface tension makes the curved sections of the free surface more concave.

4 Conclusion

In this paper we have developed a numerical method for the determination of the free surface under the influence of gravity, surface tension and surface friction. When the surface friction is not included then the equations which produce the fluid speed on the free surface are algebraic equations, but when the surface friction is included then the equation is an ordinary differential equation. The numerical results indicate that increasing the surface tension has the opposite effect on the free surface profile to increasing the surface friction. The surface tension makes the curved profile of the free surface change more sharply, whereas the surface friction smoothens out the free surface profile, namely makes the curved profile change more gently.

References

[1] Watters, G. Z. and Street, R. L., Two-dimensional flow over sills in open channels, *J. Hydraulics. Div., ASCE*, **90**, pp. 107-140, 1964.

[2] Varoglu, E. and Finn, E. D. L., Variable domain finite element analysis of free surface gravity flow. *Computers and Fluids*, **6**, pp. 103-114, 1978.

[3] Betts, P. L., A variational principle in terms of stream function for free surface flows and its application to the finite element method. *Computers and Fluids*, **7**, pp. 145-153, 1979.

[4] Belward, S. R. and Forbes, L. K., Fully non-linear two-layer flow over arbitrary topography, *J. Engng Math.*, **27**, pp. 419-432, 1993.

[5] Wen, X., Ingham, D. B. and Widodo, B., The free surface fluid flow over a step of an arbitrary shape in a channel, *Engineering Analysis with Boundary Elements*, **19**, pp. 299-308, 1997.

[6] Lonyangapuo, J. K., Elliott L. Ingham, D. B. and Wen, X., Retrieval of the shape of the bottom surface of a channel when the free surface profile is given, *Engineering Analysis with Boundary Elements*, **23**, pp. 457-470, 1999.