Outflow conditions in human arterial flow
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Abstract

This paper presents a one-dimensional model, based on Navier Stoke’s equations, that predicts the blood pressure and flow in the larger arteries. The outflow condition is based on the assumption of a constant peripheral resistance, i.e. that the ratio between pressure and flow is constant. Simulations are carried out for two cases: 1 – the aorta is regarded as a single tapered tube, and 2 – the iliac bifurcation and part of the femoral arteries are included.

1 Introduction

The major purpose of this study is to investigate the blood pressure and flow in aorta with special regard to the influence of the outflow boundary condition. First, we experiment with the aorta regarded as a single tapered vessel with the outflow boundary condition placed at the iliac bifurcation. We study how a change in the peripheral resistance in the outflow boundary condition affects the pressure profile. Second, we introduce the iliac bifurcation and move the boundary condition to the femoral arteries. Pressure profiles arising from this geometry are then compared to the previous experiments.


Originally, the main attention was directed at linearised wave propagation,
which, however, unfortunately does not capture the non-linear effects of flow-patterns, such as the increased steepening of the pressure profile along the arterial conduit. Further work taking these non-linear effects into account was therefore initiated by Anliker and Stettler, among others. In particular the work presented in Anliker[1], Anliker[2] and Stettler[14] establishes the basis for our work. 

In order to be able to deal with a continuous model not restricted to a linear system, we make a one-dimensional model based on Navier Stoke’s equations. This model consists of the aorta, the iliac bifurcation, and part of the femoral arteries, all regarded as compliant and tapered tubes. The blood is handled as an incompressible inviscid fluid, and the complete description of the system contains a momentum equation, a continuity equation, and a state equation relating the pressure at any point to the cross-sectional area. Since this study concentrates on the iliac bifurcation, we disregard all the small arteries, the capillaries, and the remaining major bifurcations. Several studies have shown that the viscoelastic behavior of the wall is important in order to have sufficient damping in the system, Holenstein[3] and Niederer[7]. However, since our studies concentrate on the effect of the distal boundary condition, we have chosen to regard the vessel walls as elastic.

2 The Model

The momentum equation consists of Euler’s equation extended by a Poiseuille friction term

$$\frac{\partial Q}{\partial t} + \frac{\partial}{\partial x} \frac{Q^2}{A} + \frac{A}{\rho} \frac{\partial p}{\partial x} = -\frac{8\pi \mu Q}{\rho A}$$ (1)

where $Q(x, t) = u(x, t)A(x, t)$ is the flow, $u(x, t)$ is the mean velocity, $A(x, t)$ is the cross-sectional area, $p(x, t)$ is the mean pressure, while $\rho$ and $\mu$ (both constant) are the density and viscosity of the blood, respectively. The continuity equation is

$$\frac{\partial A}{\partial t} + \frac{\partial Q}{\partial x} = -\Psi$$ (2)

where $\Psi$ is an outflow seepage along the arteries. The state equation defined according to linear elasticity theory, Mazumdar[5], is

$$p = \frac{ED}{r} \left(1 - \sqrt{\frac{A_0}{A}} \right) + p_0$$ (3)

where $p_0$ is the pressure of the surroundings, $r(x) = ae^{-bx}$ is the radius of vessel considered, $A_0(x) = \pi r^2(x)$ is the cross-sectional area at the pressure $p_0$, and $ED(x)$ is Youngs modulus times the wall thickness.

When modeling a bifurcation one may use the approach suggested by Stettler[14] and introduce a separate segment containing the branch (see figure 1.A). Within this segment the tube is assumed to taper linearly (as opposed to the exponential tapering in the bulk of the arteries). In order to predict the flow and the
pressure across such a segment, the following conditions are applied: First, we assume that the flow is continuous over the segment, hence

\[ Q_1 = Q_2 + Q_3 + \frac{d}{dt} \int_{x_1}^{x_2} A \, dx \]  

Second, we apply Bernoulli's relation

\[ p_1 - p_j = \frac{\rho}{2} \left( u_j^2 - u_1^2 \right) + \rho \int_{x_1}^{x_2} \frac{\partial u}{\partial t} \, ds + \Delta p_j \]  

where \( j = 2, 3 \) and \( \Delta p_j \) reflects the curvature of the branching vessel as well as the friction loss due to viscosity. The subscripts refer to the numbers shown in figure 1. However, as demonstrated in Lighthill[4], the error in assuming that the total flow into a bifurcation equals the outflow, and that the pressure is continuous, can be estimated by regarding the magnitude of the rate of change of the pressure over a bifurcation using a linearised model. For a bifurcation such as the iliac where the dimension of the branches is of order 1 cm, the error is approximately one percent, so it makes sense to regard the bifurcation as a point over which the pressure and flow are continuous (see figure 1.B). Hence, we are left with the conditions:

\[ Q_1 = Q_2 + Q_3 \quad \text{and} \quad p_1 = p_2 = p_3 \]  

In case of a symmetrical bifurcated vessel, this reduces to \( Q_1 = 2Q_2 \) and \( p_1 = p_2 \).

Figure 1: In A a conical segment is introduced around the bifurcation. In B the bifurcation is determined at a point.

3 The boundary conditions

In order to ensure a reasonable cardiac ejection rate at the proximal end of the aorta, this boundary condition is given by

\[ Q(0, t) = a \sin bt \quad \text{for} \quad t \in \left[ 0, \frac{\pi}{b} \right] \mod \frac{2\pi}{b} \]  

where the constants \( a \) and \( b \) are determined by the knowledge of the cardiac output and the period of one heart-beat.
The boundary conditions at the bifurcation point (in the case where we do not know the relationship between $Q_2$ and $Q_3$) and at the bottom of the arteries will be discussed below. In our approach, we have chosen the boundary condition suggested by Anliker[1], namely

$$Q_3 = \frac{p_1 - \text{p}_{\text{cap}}}{R}$$

(8)

where $\text{p}_{\text{cap}}$ is the capillary pressure and $R$ is the peripheral resistance. The problem with this approach is the assumption that the peripheral resistance is constant. This is not strictly physiologically true, but it is difficult to determine $R$ as an explicit function of time. In fact, Peskin argues that the pressure is approximately proportional to the flow far away from the heart, but it is also true that there is a lack of such condition near the heart, Peskin[11]. Therefore, and since the model is very sensitive to this choice, it should be investigated further. In Anliker[2] and Stettler[14] another approach is presented: Here the total outflow is

$$Q_3 = R_p(p_1 - \text{p}_{\text{cap}})A_3^2 + R_qQ_1$$

(9)

where $R_p$ and $R_q$ are lumped parameters. The first term represents outflow from relatively small rigid branches perpendicular to the main conduit, and consequently the outflow is assumed proportional to a Poiseuille flow. The second term represents outflow from a branch into two symmetrical vessels. Equation (9) can be validated empirically, but it is not possible to establish it from the physical theory. Because the outflow in general depends on both $Q_1$ and $p_1$ one may argue in favour of the linear relationship in (9).

Since we concentrate on boundary conditions that can either be derived from the theory or be empirically established, we have chosen to focus on the quite simple condition stated in (8), knowing that this condition does not mathematically quite fulfil the demands described above.

4 Model evaluation and results

The equations are solved numerically using Lax Wendroff’s two step method (Richtmayer’s version), Peskin[11], and we present simulations of two systems: System 1 constitutes only of the aorta (part I in figure 1.B) while system 2 comprises both the aorta, the iliac bifurcation and one of the femoral arteries viewed as a symmetrical bifurcation point (parts I and II in figure 1.B). Note that the boundary condition using the peripheral resistance $R$ is in both cases applied only once at the right boundary. However, when comparing $R$ from the two systems we get the relation $R_1 \approx \frac{R_2}{2}$ where $R_1$ and $R_2$ are the resistances in system 1 and 2, respectively.

As seen in figure 2 and 3 both systems show an increase of pressure in aorta (in figure 3 the aorta is the part between $x = 0$ cm and $x = 42.5$ cm). Both systems also show an increase in the steepness of the pressure profiles along the aorta. However, this is difficult to see on the 3D plots presented. In the physiological
situation it is also possible to detect a dicrotic notch and as indicated in figures 2 and 3 this is due to the reflections in the system, which in turn are consequences of the tapering of the vessels. It is seen that introduction of the bifurcation does not change these phenomena and we therefore conclude that our model shows the right qualitative behaviour.

Figures 4.A and B show a plot of $p(x_L, t)$, where $x_L$ is at the bottom of aorta, for a number of peripheral resistances. In both cases the pressure increases with increased resistance, but in case of system 1, where no bifurcations are taken into account, the reflections become more evident and also higher order reflections appear as $R$ is increased causing a large disturbance of the pressure profile. Finally, figure 4.C shows the systolic and diastolic pressures as functions of the peripheral resistance (the two top lines concern the systolic pressure and the two bottom lines the diastolic pressure). Also, in this figure it is possible to detect a more stable increase of $p(x_L, t)$ with increasing $R$.

## 5 Conclusion and further work

The most important point is the qualitative effect of moving the boundary condition to the femoral arteries. The fact that the pressure profiles in aorta are more stable when the boundary condition is placed after the iliac bifurcation, confirms the point stated in Peskin[11] that the flow and pressure are proportional only when far away from the heart. We will therefore conclude that if a suitable boundary condition should be applied (in system 1) at the bottom of aorta, the peripheral resistance should be a function of time. However, since this can be a difficult task, it is more reasonable – and also adequate – to include at least one of the major bifurcations in order to work with the boundary condition suggested in (8).

Another point we want to consider is the range of pressures in system 1. The current choice of parameters seems to create an excessive gap between the systolic and diastolic pressures. In the case of system 2 with $R = 0.5$ this gap becomes more reasonable, and we believe that this situation could be improved further through slight changes in the geometry.

These studies have been very useful to our work in estimating the level of detail needed in order to construct an adequate model for the arterial system, and we conclude that trying to lump everything into one single tapered tube is not possible.

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Figure 2: The pressure in aorta as a function of time and space.

Figure 3: The pressure in aorta ($x \leq 42.5$ cm) and in part of the femoral arteries ($x > 42.5$ cm) as a function of time and space.
Figure 4: A and B show the pressure as a function of time for four values of $R$. Since a symmetrical bifurcation is considered the value of $R$ in A equals $R/2$ in B. In C the systolic and diastolic pressures are plotted as functions of $R$ for system 1 and 2, respectively.
References


