Iterative coupling between the MFS and Kansa's method for acoustic problems

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Abstract

In the present work, a numerical frequency-domain model based on the joint use of two distinct meshless methods (the Method of Fundamental Solutions and Kansa's Method) is discussed. In this context, the MFS is used to model the homogeneous part of the propagation domain, while the Kansa's Method is applied to model the presence of possible heterogeneities. For Kansa's Method, the MQ RBF is used, and the optimal value of its free parameter is computed by minimizing the residual of the PDE throughout the subdomain. The coupling between the two parts of the propagation domain is performed iteratively, allowing totally independent spatial discretizations to be used for each of the sub-domains of the model. Given this strategy, the use of matching collocation points at common surfaces is not necessary. To improve the behavior of the iterative process, an optimized algorithm, based on the use of a varying relaxation parameter, is used to speed up and/or to ensure the convergence of the iterative coupling. A set of numerical results is here presented to illustrate the behavior of the proposed strategy in terms of convergence.
Keywords: MFS, Kansa's method, acoustics, iterative coupling.

1 Introduction

Wave propagation in fluids and solids has been thoroughly investigated in many branches of engineering and sciences, such as acoustics, geophysics or

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oceanography. Models based on the Boundary Element Method (BEM), Finite Element Method (FEM) and Finite Difference Method (FDM) have been extensively used in the study of these phenomena during the last decades. The BEM, in particular, has proved to be a valuable tool in this analysis, since it provides accurate results, while allowing for a simple description of the system, requiring only its boundaries to be discretized [1, 2]. However, the BEM formulation requires the knowledge of fundamental solutions for each region of the problem, which can only be defined for simple configurations of the host medium; additionally, the BEM also requires that a number of boundary integrations is performed, some of them involving singular functions.

Meshless methods have emerged in the 1990s, and their main feature is that they do not require explicit domain or boundary discretization. Examples of such techniques are the Method of Fundamental Solutions (MFS) [3–5] and the Radial Basis Functions (RBF) collocation method (Kansa's Method [6, 7]). The MFS is mathematically simple, and is also based on the prior knowledge of the Green's functions of the propagation sub-domains. As with the BEM, this limitation poses problems whenever inhomogeneous domains are to be analyzed. The Kansa's Method (KM), on the other hand, follows a different approach, reproducing the solution within a specific sub-domain as a linear combination of RBFs, and so not requiring the use of Green's functions. Although different studies have been published on the use of these techniques in acoustics, they are mostly restricted to solve the Helmholtz equation in problems involving 2D and 2.5D domains composed of homogeneous sub-domains.

A possible strategy to the solution of problems with point-to-point variations of properties is to use FEM or FDM models, which are usually adequate for this purpose. However, they pose problems when dealing with infinite or semi-infinite (halfspace) domains, since they require the use of virtual boundaries with a special mathematical treatment. In this work, an alternative approach to analyse infinite systems, with localized heterogeneities, is proposed. A combination of two meshless methods is here considered, namely the MFS and the KM, whereas the MFS is used to simulate the infinite part of the domain, and the KM to model the heterogeneity. An iterative coupling approach is used to handle the combination of these two different numerical procedures. In fact, the use of an iterative coupling methodology exhibits several advantages when compared to direct coupling schemes, such as: (i) sub-domains can be analysed separately, leading to smaller and better-conditioned systems of equations (different solvers, suitable for each sub-domain, may be employed); (ii) independent discretizations may be considered for each sub-domain, allowing non-matching nodes on common interfaces to be easily considered; (iii) only interface routines are required when one wishes to use existing codes to build coupling algorithms.

The present work is organized as follows: first, the governing equations of the physical problem are generically presented; then, the two meshless methods (MFS and KM) are described; there follows a description of the iterative coupling processes, including a description of the optimisation methodology. Finally, a numerical example is presented, illustrating the accuracy, performance and potentialities of the proposed procedure.

2 Mathematical formulation

In the frequency domain, the propagation of sound waves in a fluid medium, with density ρ and allowing a sound speed α, assuming null initial conditions, is governed by the Helmholtz equation

$$\nabla^2 p + \left(\frac{\omega}{\alpha}\right)^2 p = 0 \tag{1}$$

where p is the sound pressure within the fluid. The solution of this equation can only be obtained analytically for very simple configurations, while for more generic cases the use of numerical techniques becomes necessary.

2.1 Formulation of the Kansa's method for interior problems

Let $\{(x_k, y_k)\}_{k=1}^{N}$ be N distinct collocation points in which $\{(x_k, y_k)\}_{k=1}^{NI}$ are interior points and $\{(x_k, y_k)\}_{k=NI+1}^{N}$ are boundary points. A typical distribution of interior and boundary collocation points is given in Figure 1 for the case of a circular domain.

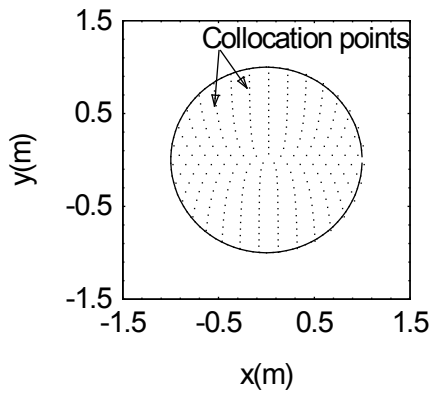

Figure 1: Typical collocation point distribution for a circular fluid filled domain.

In Kansa's method, the exact solution p, in equation (1), is approximated as $\hat{p}$ by

$$\hat{p}(x, y) = \sum_{k=1}^{N} a_k \varphi_k(x, y) \tag{2}$$

where $\varphi_k(x,y)$ are radial basis functions and $\left(a_k\right)_{k=1}^{N}$ are coefficients to be determined (N unknowns). Although there are many types of radial basis functions available, MQ is the most widely adopted RBF in Kansa's method, and can be defined as

$$\varphi_k(x,y)=\sqrt{r_k^2+c^2} \text{ with } r_k=\sqrt{\left(x-x_k\right)^2+\left(y-y_k\right)^2} \tag{3}$$

where c is a shape parameter. The parameter c can greatly affect the accuracy of the approximation, but the determination of an optimal c is still a research problem (see, for example, Cheng [8]). In a recent paper by Godinho and Tadeu [9], the definition of this parameter is performed by choosing the value that allows minimizing the residual of the PDE throughout the analysis domain. Here, we follow this strategy for the definition of the shape parameter.

Using this approximation, writing equation (2) for each interior point and imposing the required boundary conditions at boundary points, N linear equations are defined. It is important to note that, given the nature of this method, it becomes simple to ascribe different properties to each of the collocation points, and thus it is straightforward to model media with point-to-point variations.

The resolution of the final linear system of equations (NxN) allows the unknowns $\left(a_k\right)_{k=1}^{N}$ to be computed. The approximate solution at any point in the interior domain can then be obtained using equation (2).

2.2 Formulation of the MFS for external problems

The MFS approximates the solution in terms of a linear combination of fundamental solutions for the governing equation. For this case, if an external problem is considered, these fundamental solutions are centered on NS virtual sources, placed outside the domain of interest over a fictitious boundary, as illustrated in Figure 2.

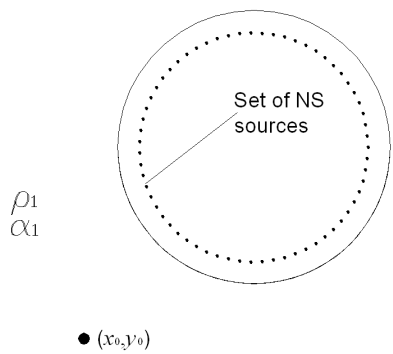

Figure 2: Typical collocation point and virtual source distribution for a fluid domain with a circular inclusion, illuminated by a source at $\left(x_0,y_0\right)$.

In the presence of a linear source positioned at (x_0, y_0), the approximate MFS solution can then be written as

$$\hat{p}(x,y) = \sum_{k=1}^{NS} \left[b_k G(x, x_k, y, y_k) \right] - \frac{\mathrm{i}}{2} \mathrm{H}_0^{(2)} \left(\frac{\omega}{\alpha} \sqrt{(x - x_0)^2 + (y - y_0)^2} \right) \tag{4}$$

where $\{(x_k, y_k)\}_{k=1}^{NS}$ are *NS* distinct source points on a fictitious boundary as shown in Figure 2, b_k is the unknown amplitude of the k^{th} source, and $G(x, x_k, y, y_k) = -\frac{\mathrm{i}}{4} \mathrm{H}_0^{(2)} \left(\frac{\omega}{\alpha} \sqrt{(x - x_k)^2 + (y - y_k)^2} \right)$ are the Green's functions for acoustic wave propagation in a homogeneous fluid. The fictitious boundary is placed outside the domain of the problem to avoid singularities in the analysis domain. In this work the $\{(x_k, y_k)\}_{k=1}^{N}$ source points are chosen *a priori* using a fixed scheme. As the geometry of the problem is circular, the source points are thus equally spaced along a circle concentric with the inclusion.

3 Iterative coupling

The iterative coupling between the MFS and Kansa's method is performed by means of a so-called Neumann-to-Dirichlet coupling strategy, in which the following analysis sequence is performed:

- First, the MFS sub-domains are analysed, by prescribing the values of the normal velocities along the boundary. As an initial approximation to start the iterative process, null velocities may be prescribed;
- After computing the unknown amplitude coefficients for the MFS, the pressure values at the boundary points of the interior sub-domains (to be analysed using Kansa's method) are calculated;
- The pressure values evaluated at the boundary nodes are then prescribed for the interior problem, and Kansa's method is then applied to find a solution for the interior domain;
- Using Kansa's method, new values of the normal velocity are then computed at the MFS boundary points, and a new estimation of the velocities for the external problem is then computed.
- The iterative process then goes back to the initial step, and proceeds until convergence is reached.

In the proposed iterative algorithm, a relaxation parameter is used to compute the normal velocities at the MFS boundary for each iterative step; mathematically, the application of this parameter may be written as

$$\mathbf{V}^{(k+1)} = (\lambda)\mathbf{V}^{(k+\lambda)} + (1 - \lambda)\mathbf{V}^{(k)} \tag{5}$$

where λ is the relaxation parameter, $\mathbf{V}^{(k+1)}$ is the velocity to consider at the next iteration, $\mathbf{V}^{(k+\lambda)}$ is the velocity calculated through Kansa's method at the end of iteration k.

In order to evaluate an optimal relaxation parameter, the following square error functional is here minimized:

$$f(\lambda) = \| \bar{\mathbf{X}}^{(k+1)} - \bar{\mathbf{X}}^{(k)} \|^2 \tag{6}$$

where $\bar{\mathbf{X}}$ is related to the MFS prescribed values at the common interfaces (i.e., it represents the prescribed velocities for the MFS). Taking into account the relaxation of the prescribed values for the ($k+1$) and (k) iterations, the following equations may be written for the iteration (k) and (k-1):

$$\bar{\mathbf{X}}^{(k+1)} = (\lambda)\mathbf{X}^{(k+\lambda)} + (1-\lambda)\mathbf{X}^{(k)} \tag{7a}$$

$$\bar{\mathbf{X}}^{(k)} = (\lambda)\mathbf{X}^{(k+\lambda-1)} + (1-\lambda)\mathbf{X}^{(k-1)} \tag{7b}$$

Substituting equations (7) into equation (6) one may write:

$$\begin{aligned} f(\lambda) &= \| (\lambda)\mathbf{W}^{(k+\lambda)} + (1-\lambda)\mathbf{W}^{(k)} \|^2 = \\ &= (\lambda^2) \| \mathbf{W}^{(k+\lambda)} \|^2 + 2\lambda(1-\lambda)(\mathbf{W}^{(k+\lambda)}, \mathbf{W}^{(k)}) + (1-\lambda)^2 \| \mathbf{W}^{(k)} \|^2 \end{aligned} \tag{8}$$

where the inner product definition is employed (e.g., $(\mathbf{W}, \mathbf{W}) = \| \mathbf{W} \|^2$) and new variables, as defined in equation (9), are considered.

$$\mathbf{W}^{(k+\lambda)} = \mathbf{X}^{(k+\lambda)} - \mathbf{X}^{(k+\lambda-1)} \tag{9}$$

To find the optimal λ that minimizes the functional $f(\lambda)$, equation (8) can be differentiated with respect to λ and the result is set to zero, as described below:

$$(\lambda) \| \mathbf{W}^{(k+\lambda)} \|^2 + (1-2\lambda)(\mathbf{W}^{(k+\lambda)}, \mathbf{W}^{(k)}) + (\lambda-1) \| \mathbf{W}^{(k)} \|^2 = 0 \tag{10}$$

Re-arranging the terms in equation (10), yields:

$$\lambda = (\mathbf{W}^{(k)}, \mathbf{W}^{(k)} - \mathbf{W}^{(k+\lambda)}) / \| \mathbf{W}^{(k)} - \mathbf{W}^{(k+\lambda)} \|^2 \tag{11}$$

which provides an optimal value for the relaxation parameter λ, at each iterative step. This expression is very simple to implement and does not require significant computational effort.

4 Numerical example

To illustrate the behavior of the proposed coupling strategy, we consider a test problem in which a line source is positioned at coordinates x=-5.0 m and y=0.0 m, within a fluid medium allowing a velocity of 1500 m/s; within this medium, at the origin of the axis system, a circular inclusion of radius 1.0 m is

considered, filled with a different fluid which allows sound waves to propagate at 2000.0 m/s. Both the host medium and the inclusion exhibit a density of 1000 kg/m^3. To model this problem, the MFS is used to account for the outer infinite medium, while Kansa's method is used to model the inclusion.

In Figure 3, the results calculated along the interface between both methods in terms of acoustic pressure for three different frequencies are illustrated. The chosen frequencies correspond to 250 Hz, 1000 Hz and 2000 Hz, and the results

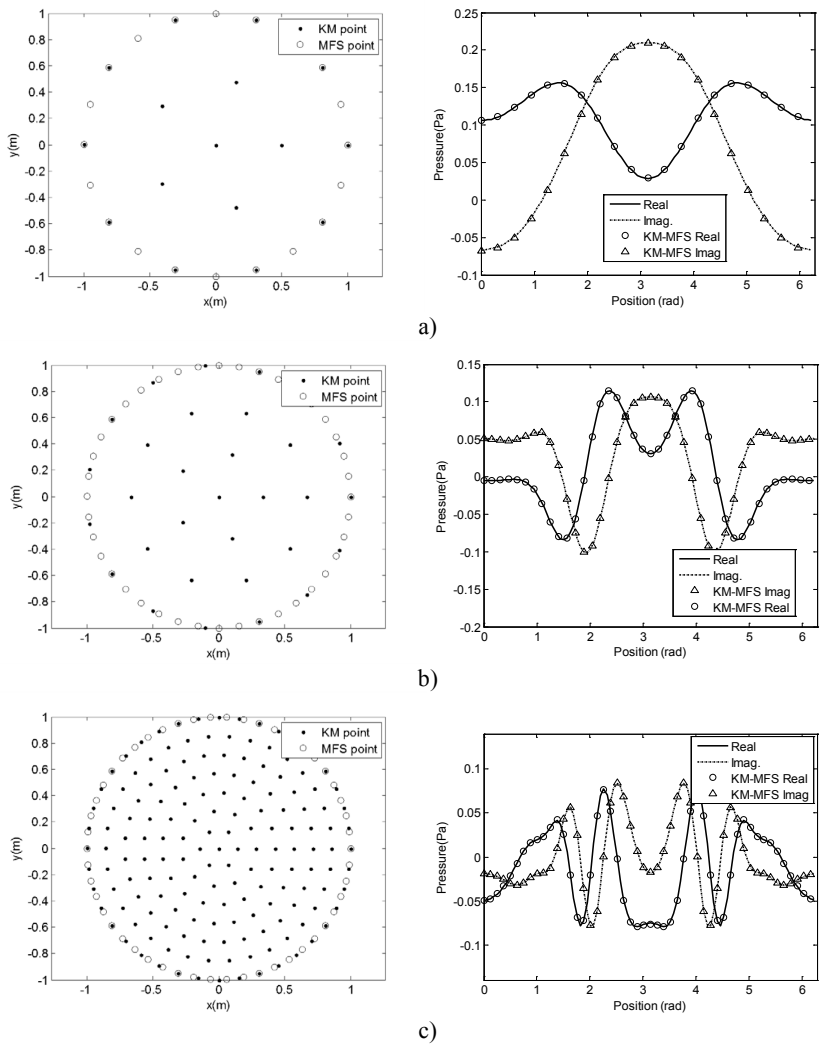

Figure 3: Point distribution (left) and pressure results (right) for specific frequencies: a) 250 Hz (14 iterations); b) 1000 Hz (26 iterations); c) 2000 Hz (66 iterations).

are plotted together with the analytical solution of the problem. The number of boundary points for the MFS and for Kansa's method is progressively increased as the wavelength decreases, to allow for correctly modeling the problem. Even for the higher frequency, the maximum number of boundary points is of 50 for the MFS and of 40 for Kansa's method, corresponding to 10 and 6 points per wavelength, respectively. Additionally, one should note that independent descriptions of the boundary are adopted for the MFS and Kansa's method, with non-matching boundary nodes for the two methods.

The presented results evidence a very good match between the numerical and the analytical solutions, for all calculated frequencies. It should be noted that for each frequency, the free parameter adopted for Kansa's method was different, and corresponded to c=8.13, c=3.37 and c=1.71; these values were computed by minimizing the residual of the PDE throughout the domain [9], and indicate the strong dependence of an "optimal" free parameter with respect to the collocation point density and to the frequency.

Finally, in Figure 4a), the total number of iterations required for convergence for a full range of frequencies is displayed, considering a constant point distribution with 50 boundary points for the MFS and 40 boundary points for the Kansa's method. As can be seen in this figure, the number of iterations is relatively small, reaching a maximum of around 70 iterations for the higher frequencies. In Figure 4b, a similar plot is presented considering the calculation to be performed without any optimization of the relaxation parameter, and assuming this parameter to be constant, and equal to 0.5. As can be seen, the number of iterations is always higher when the relaxation parameter is not optimized, and it even occurs that, for a considerable number of frequencies, convergence is not reached. This clearly shows the importance of adopting this optimized strategy for the coupling between the two methods.

5 Final remarks

In this work, a numerical frequency-domain model, based on the coupling between the MFS and Kansa's Method has been presented and discussed. The MFS has been used to model the external infinite domain, while Kansa's Method was used to model a localized heterogeneity. The coupling between the two parts of the propagation domain was performed iteratively, based on a Neumann-to-Dirichlet approach with an optimized relaxation parameter, allowing independent spatial discretization between the different sub-domains of the model. The application of the method was illustrated for a simple example, revealing that the methodology can be efficient; the presented example also allowed observing that it can be very important to make use of optimized relaxation parameters in order to speed up or even to ensure convergence of the coupling algorithm.

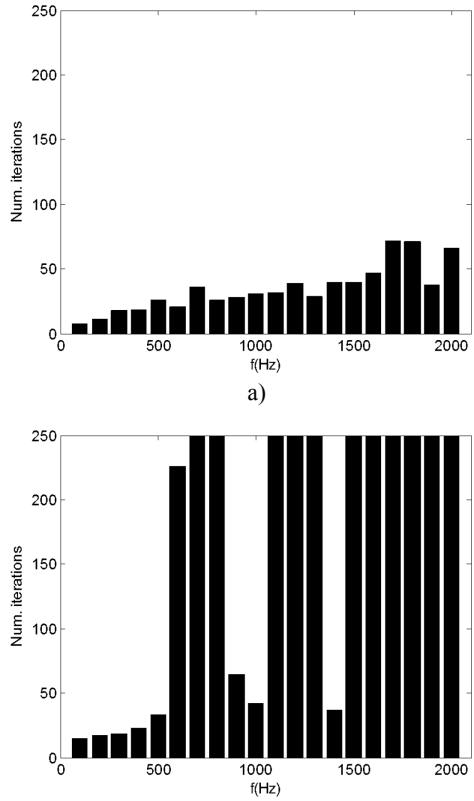

Figure 4: Number of iterations required for convergence considering optimized (a) and constant (b) relaxation parameters.

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